According to the Latest syllabus of Kashmir University

A TEXT BOOK
OF
ELEMENTARY CALCULUS

For 1st year of Three-year Degree Course

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PREFACE TO THE THIRD EDITION

The present edition has enabled me to completely overhaul the chapter on Integration besides revising the book and adding to it Chapter IX on the "Statement of Maclaurin's Theorem and its applications to Simple Expansions." Up-to-date University papers have also been added to enable the student to know what sort of questions are usually set at the university level.

It is, therefore, hoped that the book in its present form will prove more useful to the students than before.

Srinagar

February, 1978 **G.M. Shah**

PREFACE TO THE SECOND EDITION

The book has been thoroughly revised and *Theory of Sets* has been included in the first chapter in accordance with the new syllabus of the Kashmir University. *Integration as the limit of a sum* has also been added to the chapter on Integration.

Srinagar

July, 1968 **G.M. Shah**

PREFACE TO THE FIRST EDITION

The chief aim in writing this book has been to provide a suitable text-book on Calculus for the use of students studying in the first year of three-year degree course. It has been written in strict accordance with the syllabus prescribed by the J. & K. University. Every attempt has been made to make the concepts clear. Accordingly, all important topics such as "Limits", "Differentiation", etc., have been dealt with in detail, and special care has been taken to deal with the difficulties of students of average ability. Various articles have been explained and illustrated by means of a number of solved examples. In short, no pains have been spared to present the subject matter in as lucid a manner as possible.

Suggestions for the improvement of the book shall be thankfully received and acknowledged.

Srinagar

August, 1965 **G.M. Shah**

SYLLABUS OF THE UNIVERSITY OF KASHMIR

For

1st Year T.D.C.

MATHEMATICS PAPER—II

For

1978 onwards

CALCULUS

Section B

(1) Elementary Functions such as :—Polynomial Functions, Rational functions, Trigonometric and inverse trigonometric functions, Exponential and logarithmic functions, Hyperbolic functions and Inverse hyperbolic functions.

Section C

(1) Intuitive notion of the limit of a function.

(2) Derivative Algebra and derivable functions.

(3) Differentiation of algebraic, trigonometric, inverse trigonometric, exponential, logarithmic and hyperbolic functions.

(4) The derivative as rate of change.

(5) Dervative as tangent-slope. Applications to tangents and normals in Cartesian co-ordinates.

Section D

(1) Successive differentiation. Leibnitz rule.

(2) Statement of Maclaurin's Theorem with applications to simple expansions.

(3) Definition of integral as limit of a sum with very simple applications.

CONTENTS

1

Conception of a Function

1·1. Definitions

- Quantity
  - Variable
    - Related (or dependent) variable (or function)
    - Independent variable
  - Constant
    - Absolute constant
    - Arbitrary constant

1. **Quantity**. Anything which can be measured or which is capable of being added, subtracted, multiplied, or divided is called a *Quantity*. Distance, force, area, etc., are examples of quantities.

Quantities are divided into two classes, *viz.*, variables and constants.

2. **Constant**. A constant is a quantity which retains the same value in an investigation. For instance, 1, 2, 3......... *a*, *b*, *c*......... are constants. Constants are further sub-divided into (*i*) *Absolute constants*, and (*ii*) *Arbitrary constants*.

(*i*) *Absolute constants.* These are the constants which do not change at all with the change in time, space, place etc. For instance, 1, 2, 3......... are absolute constants.

(*ii*) *Arbitrary constants.* These are the constants which change their value with the change in time, space, or place etc. These are generally denoted by the *first letters* of alphabets, *i.e.*, *a*, *b*, *c*.........

3. **Variables**. A variable is a quantity which changes its value in an investigation. For example, the height of the sun above the ground at different hours of the day, the temperature of a place on different days of the year, are variable quantities.

Variables are generally denoted by ***the last letters*** of the alphabet, *i.e.* u, v, w, x, y, z........

4. **Function.** If two variables are so related that the value of the one depends upon that of the other, the two variables are called *Related variables*. In such a case, the former is said to be the *function* of the latter.

Illustrations

(*i*) Let us consider the expression x^2+2x+1. For different values of x, this expression will assume different values. Let us put $y=x^2+2x+1$, then the different values of y corresponding to the different values of x are given below :

$$x=1 \qquad y=4$$
$$x=0 \qquad y=1$$
$$x=3 \qquad y=16 \text{ etc.}$$

Evidently, the values of y depend upon the values of x. Therefore, we can say that y is a *function* of x, while x is known as the *argument*. This relation is *symbolically* expressed as

$$y=f(x).$$

(*ii*) Consider $\quad y=\tan x$

If $\quad x=\dfrac{\pi}{4} \qquad y=1$

if $\quad x=\dfrac{\pi}{3} \qquad y=\sqrt{3}$ and so on.

Here also we find that the value of y or $\tan x$ ***depends*** upon the value of x. Hence $y=\tan x$ is a function of x.

Note. (*i*) Different symbols such as $f(x)$, $F(x)$, $\phi(x)$, $\psi(x)$, etc., are used to represent functions when more than one function are under consideration.

(*ii*) A function of more than one variable is denoted as $f(x, y)$, $F(x, y)$, $f(x, y, z)$, etc.

1·2. If $f(x)$ be a function of x, we can find its value for a given value of the argument x. The value of $f(x)$ for $x=a$ is denoted by $f(a)$.

Thus, if $f(x)=x^2+1$, we have

$$f(a)=a^2+1 \qquad \text{for } x=a$$
$$f(1)=2 \qquad \text{for } x=1$$
$$f(0)=1 \qquad \text{for } x=0$$

and $\quad f\left(\dfrac{1}{x}\right)=\dfrac{1}{x^2}+1$

when x is replaced by $\dfrac{1}{x}$.

1·3. Geometrical Representation of a Function

The relation between a function and its argument is *geometrically* represented by a curve. This curve is called the *graph* of the said function.

Let $y=f(x)$ be a function of x. We can get different values of y corresponding to different values of x. Let us take the values of x as the abscissae and those of y as the ordinates and plot the points. Then a curve through these points is known as the *graph* of $y=f(x)$.

Example. Draw the graph of

$$y=3x+5.$$

Sol. The table of values of x and y is given below :

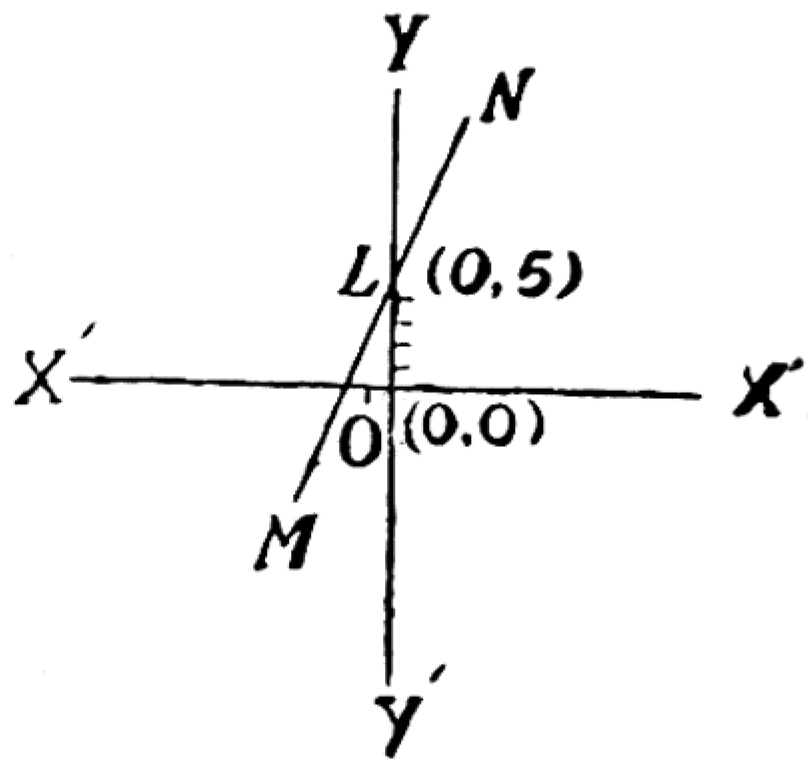
$x=$	0	-1	1	-2	-4	-3
$y=$	5	3	8	-1	-7	-4

The graph is a straight line which has been obtained by plotting the points

(0, 5), (-1, 2)

(1, 8), (-2, -1)

(-4, -7) and (-3, -4)



Exercise I

Draw the graphs of the following functions :

1. $y^2=8x$.
2. $y=\sin x$.
3. $y=\cos x$.
4. $y=x+8$.
5. $y=\sqrt{25-x^2}$.

1·4. Classification of Functions

(*a*) *Algebraic and Transcendental functions.* An algebraic function is one which consists of a finite number of terms involving six operations of algebra, *viz.*, addition, subtraction, multiplication, division, involution and evolution.

Thus x^2+5x+1, $(3x^2+8x+2)(3x-1)$, $\dfrac{6x+2}{x^3+1}$, $\dfrac{\sqrt{x^2+1}}{(2x+3)^{1/2}}$

etc., are all algebraic functions. Any function which is not algebraic is called a *Transcendental* function. Transcendental functions are further classified as under :

(*i*) The *Trigonometric* functions such as $\cos x$, $\operatorname{cosec} x$, etc.

(*ii*) The *Inverse Trigonametric* functions such as $\tan^{-1} x$, $\sin^{-1} x$, etc.

(*iii*) The *Logarithmic* functions such as $\log_a x$, $\log_a (1+x)$, etc.

(*iv*) The *Exponential* functions such as a^{mx}, e^{5x}, etc.

(*v*) The *Incommensurable* powers of a variable such as $x^{\sqrt{2}}$, $x^{\sqrt{3}}$, etc.

(*b*) *Explicit and Implicit functions.* A variable y is said to be an *explicit function* of another variable x, when, in an equation, expressing the functional relation, y occurs *singly and alone on the left hand-side* of the equation, and does not occur on the right-hand side. Thus in the equations

$$y=3x^2+5x+1, \quad y=\sin x+\cos x$$

y is an explicit function of x.

When y occurs *mixed up* with x on one or both sides of the equation, y is said to be an implicit function of x. Thus in the equation

$$x^3+y^3=3axy$$

y is an implicit function of x.

(*c*) *Single-valued and Many valued functions.* y is said to be a *single-valued* function of x, if corresponding to *one* value of x, there exists *one and only one* value of y as in

$$y=3x+4 \text{ etc.}$$

y is said to be a *many-valued* function of x, if corresponding to *one* value of x, there exist *more than one* values of y, as in

$$y^2=4x,\ y^3=8x,\ y=\sin^{-1} x, \text{ etc.}$$

(*d*) *Even and Odd Functions.* A function $f(x)$ is said to be an *even function* of x, if $f(-x)=f(x)$. For example $f(x)=\cos x$ is an even function of x, for

$$f(-x)=\cos(-x)=\cos x=f(x).$$

It is said to be an *odd function* of x, if $f(-x)=-f(x)$. For example $f(x)=\sin x$ is an odd function of x, for

$$f(-x)=\sin(-x)=-\sin x=-f(x).$$

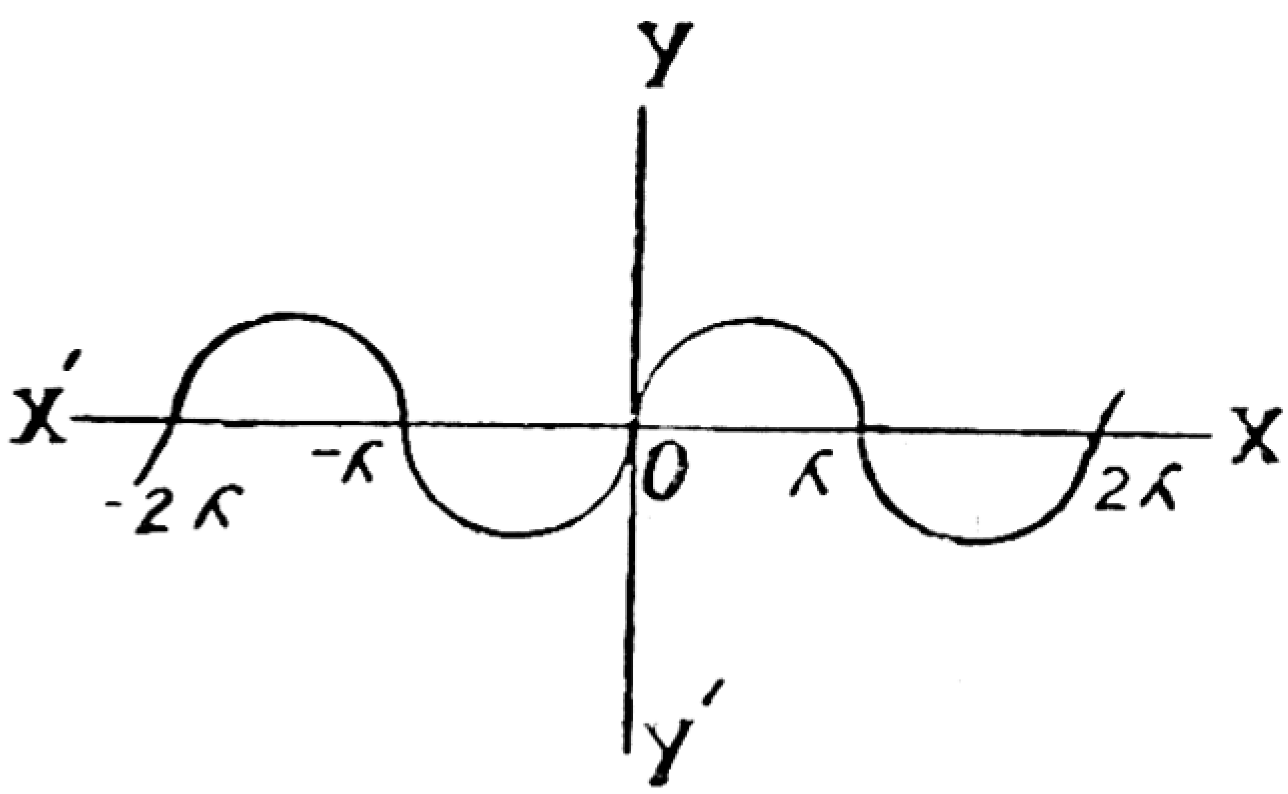
(*e*) *Periodic functions.* If $f(x)=f(x+a)$ where "a" is a constant, then $f(x)$ is said to be a *periodic function* of x and "a" is called the period.

For instance, in $f(x) = \tan x$,

$$f(\pi + x) = \tan(\pi + x) = \tan x = f(x).$$

$\therefore$ $\tan x$ is a periodic function with π as the period.

(*f*) *Continuous and Discontinuous Functions.* A function is said to be *continuous* or *discontinuous* according as its graph is continuous or discontinuous (without gap or break). For instance, the function $y = \sin x$ is continuous, for its graph is continuous. This graph is given below :

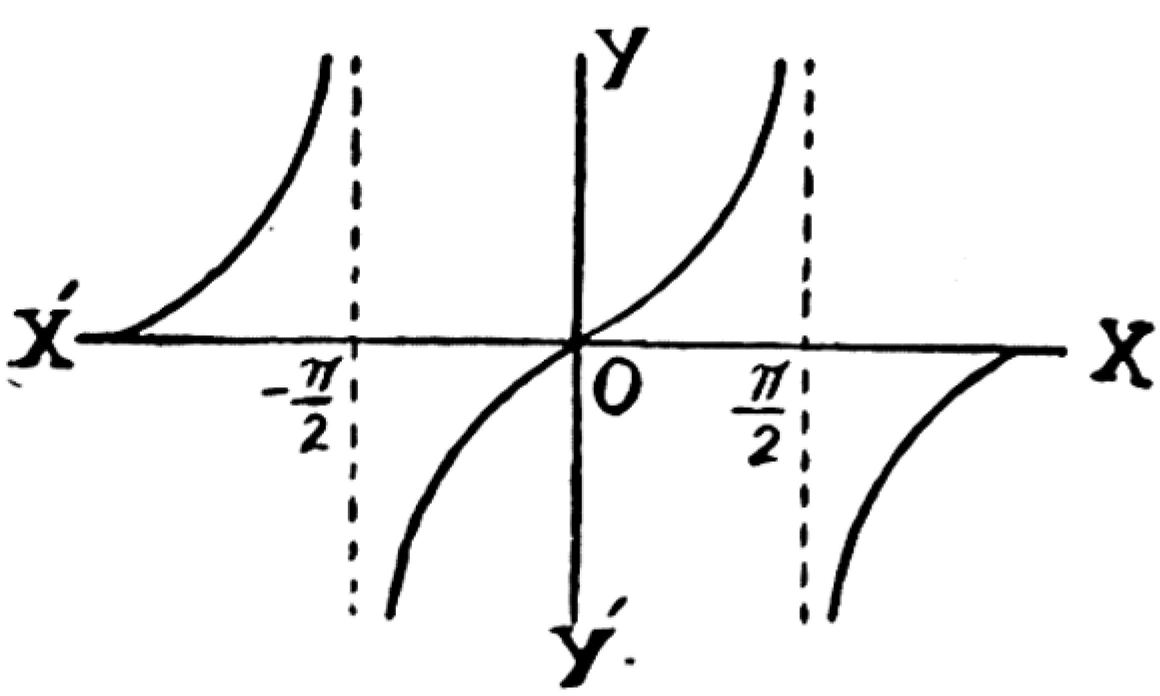


Similarly, the function $y = \cos x$ is also continuous.

The function $y = \tan x$, is however, discontinuous because its graph is discontinuous. There is a break at $x = \frac{\pi}{2}$ and also at

$$x = -\frac{\pi}{2}.$$

The graph is given bolow :



Exercise II

1. Define a function. State various kinds of functions, giving examples in each case.

2. If $f(x)=x^2+2x-5$, find $f(2), f(-1), f(0), f\left(\frac{1}{x}\right), f(a+h)$.

3. If $f(x)=\sin x$, find $f\left(\frac{\pi}{2}\right), f\left(\frac{\pi}{3}\right), f(0)$.

4. If $f(x)=\tan x$, show that

$$f\left(\frac{\pi}{4}-x\right)=\frac{1-f(x)}{1+f(x)}.$$

5. If $f(x)=\log_a x$, show that

$$f(mn)=f(m)+f(n).$$

6. If $\phi(x)=\frac{1-x^2}{1+x^2}$, show that

$$\phi(\tan\theta)=\cos 2\theta.$$

7. If $f(x)=x^2$, find $\{f(a+h)-f(a)\}/h$.

8. If $f(x)=\tan x$, show that

$$\frac{f\left(\frac{\pi}{4}+h\right)-f\left(\frac{\pi}{4}\right)}{h}=\frac{2\tan h}{h(1-\tan h)}.$$

9. Prove that $\cos x+\sec x$ is an even function.

10. Show that $f(x)=5x^5+7x^3+x$ is an odd function.

11. If $y=f(x)=\frac{3x-1}{4x-3}$, show that $x=f(y)$.

12. If $f(x)=2x\sqrt{1-x^2}$, show that

$$f\left(\sin\frac{x}{2}\right)=\sin x.$$

13. If $f(x)=\frac{e^x+1}{e^x-1}$, prove that

$$f(-x)=-f(x).$$

14. If $e^x+e^{-x}=y$, show that

$$x=\log\left[\frac{y\pm\sqrt{y^2-4}}{2}\right].$$

15. For what values of x, have the following functions no definite value :

(*i*) $\frac{1}{x-1}$, (*ii*) $\frac{1}{x^2-5x+6}$, (*iii*) $\frac{1}{\sin x}$?

16. Show that $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$ are periodic functions. What are their periods ?

1·5. The phrase "x tends to a".

Let x pass successfully through an infinity of values according to some law such that for each value taken by x, we can distinguish the values which *precede* from those which *follow*, and that no value of x is the last. *If now the successive values of x approach a definite number "a" in such a way that the numerical value af $x-a$ becomes and remains smaller than every given positive number, however small, then we say that x tends to "a" (written as $x \to a$ or x has the limit "a").*

It should be *clearly* understood that the number "a" itself may or may not be a value of x. If x takes on *successively* the sequence of values

$$1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots, \frac{1}{n}, \ldots\ldots\ldots \text{then Lt } x=0$$

even though 0 is not a value of x.

1·6. The phrase "x tends to infinity"

Let x take a series of values successively which ultimately become and remain greater than every real positive number, however great, then we say that "*x tends to plus infinity*" and we write it as $x \to +\infty$.

On the other hand, if the successive values assumed by x become and remain smaller than every negative number, then we say that "*x tends to minus infinity*" and we write $x \to -\infty$.

1·7. Limit of a function

Let us now consider the behaviour of a function $f(x)$ when x approaches any given value "a". As x approaches the value "a", the value of $f(x)$ may become closer and closer to a definite number l. Let us consider, for example, the function $f(x)=2x+1$, and let x approach the value 1. Then it is easy to see that as x becomes closer and closer to 1, $2x+1$ becomes closer and closer to 3. Further, the difference between $2x+1$ and 3, *i.e.* $2(x-1)$ can be made as small as we please by taking x sufficiently close to 1. In such a case we say that $2x+1$ has a limit 3 as x tends to 1, and write

$$\underset{x \to 1}{\text{Lt}}\ f(x)=3.$$

1·8. Difference between the "value" and the "limit" of a function

Let us consider the function

$$f(x)=\frac{x^2-1}{x-1}.$$

For $x=2$, $f(x)=3$, and, in general, the value of this function for $x=a$ is

$$f(a)=\frac{a^2-1}{a-1}.$$

If, however, we try to find the value of the function for $x=1$, we face some difficulty. For $x=1, f(x)=\frac{0}{0}$ which is *absurd.* Thus, we cannot find the *exact value* of this function for x *exactly equal* to 1. We can, however, find the *limit* of $f(x)$ when x approaches 1.

The *behaviour* of $f(x)$ when x approaches 1 through values less than 1 or through values greater than 1, is indicated in the table given below :

$x=$	·9	·99	·999	$\ldots \rightarrow 1 \leftarrow \ldots$	1·001	1·01	1·1
$f(x)=$	1·9	1·99	1·999	$\ldots \rightarrow 2 \leftarrow \ldots$	2·001	2·01	2·1

We notice that as x approaches 1 either from left-hand side or from right-hand side, $f(x)$ approaches 2. Hence the limit of $f(x)$ for x approaching 1 is 2. In symbols, we write all this as

$$\underset{x \rightarrow 1}{\text{Lt}} \frac{x^2-1}{x-1}=2.$$

Hence, we define the *limit* of a function as under :

If a function $f(x)$ approaches a fixed quantity "l" when x approaches "a", then "l" is said to be the limit of $f(x)$ as $x \rightarrow a$. In symbols, we write

$$\underset{x \rightarrow a}{\text{Lt}}\ f(x)=l.$$

1·9. Evaluation of the limit of a function

The limiting value of a function $f(x)$ for a given value "a" of x may sometimes be the same as its exact value for $x=a$. Hence, in order to find $\underset{x \rightarrow a}{\text{Lt}}\ f(x)$ we first simply substitute "a" for x and find $f(a)$. This will give the required limit. But if $f(a)$ assumes a form which is meaningless, then we first simplify $f(x)$ so that when we substitute "a" for x, it does not assume a meaningless form, and in this way we get the required limit.

This is illustrated below.

Example 1. Find $\underset{a \rightarrow 2}{\text{Lt}}\ (x^2+2x+4)$.

Sol. Here $\quad f(x)=x^2+2x+4$

Putting $\quad x=2$, we get

$f(2)=(2)^2+2(2)+4=12$

which is not meaningless.

Hence, $\quad \underset{x \rightarrow 2}{\text{Lt}}\ (x^2+2x+4)=12.$

Example 2. Find $\underset{x\to 2}{\text{Lt}}\dfrac{x^2-4}{x-2}$.

Sol. Let $f(x)=\dfrac{x^2-4}{x-2}$

Putting $x=2$, we find that

$$f(2)=\frac{4-4}{2-2}=\frac{0}{0},$$

which is meaningless and cannot, therefore, be the *limit* of $f(x)$. Hence, we try to simplify $\dfrac{x^2-4}{x-2}$. If $x=2$, then $x-2=0$ and we cannot divide x^2-4 by $x-2$ for division by zero is not permissible. However, in the problem x is not given to be equal to 2. It *approaches* 2, so that $x-2\neq 0$ and we *can* perform the division. Thus, the given expression

$$\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{x-2}=x+2.$$

If we now substitute 2 for x,

$$\frac{x^2-4}{x-2}=x+2$$

takes the value $2+2=4$

$$\therefore \qquad \underset{x\to 2}{\text{Lt}}\frac{x^2-4}{x-2}=\underset{x\to 2}{\text{Lt}}(x+2)=4.$$

Another Method (Most practicable)

Let $\qquad x=2+h$

Then, $\qquad h\to 0$ as $x\to 2$

$$\therefore \qquad \underset{x\to 2}{\text{Lt}}\frac{x^2-4}{x-2}=\underset{h\to 0}{\text{Lt}}\frac{(2+h)^2-4}{(2+h)-2}$$

$$=\underset{h\to 0}{\text{Lt}}\frac{4h+h^2}{h}$$

$$=\underset{h\to 0}{\text{Lt}}(4+b)=4.$$

Example 3. Evaluate $\underset{x\to\infty}{\text{Lt}}\dfrac{5x^2+7x-3}{4x^2-2x+7}$.

Sol. The symbol $x\to\infty$ indicates that x takes up values which go on continually increasing. Now, if x is large, both the numerator and the denominator of the given expression take the form $\dfrac{\infty}{\infty}$ which is meaningless. Hence, here also we have to simplify the expression before we find the limit. Thus dividing the numerator and the denominator by x^2, we have

$$\frac{5x^2+7x-3}{4x^2-2x+7}=\frac{5+\frac{7}{x}-\frac{3}{x^2}}{4-\frac{2}{x}+\frac{7}{x^2}}$$

$$\therefore \quad \underset{x\to\infty}{\text{Lt}} \frac{5x^2+7x-3}{4x^2-2x+7}=\underset{x\to\infty}{\text{Lt}} \frac{5+\frac{7}{x}-\frac{3}{x^2}}{4-\frac{2}{x}+\frac{7}{x^2}}$$

Now, as x increases, $\frac{1}{x}$, $\frac{1}{x^2}\ldots\ldots$ decrease so that we can say that as $x\to\infty$, $\frac{1}{x}$ and $\frac{1}{x^2}\to 0$.

Hence, the required limit$=\frac{5}{4}$.

Alternative Method (Less practicable)

Let $\qquad x=\frac{1}{y}$

Then $\qquad y\to 0 \quad \text{as} \quad x\to\infty$

$$\therefore \quad \underset{x\to\infty}{\text{Lt}} \frac{5x^2+7x-3}{4x^2-2x+7}=\underset{y\to 0}{\text{Lt}} \frac{\frac{5}{y^2}+\frac{7}{y}-3}{\frac{4}{y^2}+\frac{2}{y}+7}$$

$$=\underset{y\to 0}{\text{Lt}} \frac{5+7y-3y^2}{4-2y+7y^2}$$

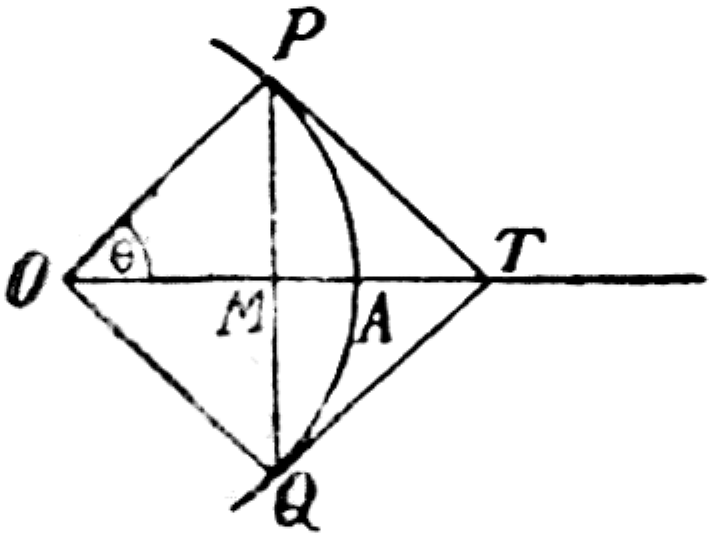
$$=\tfrac{5}{4}.$$

1·10. Some Important Limits

1. $\underset{\theta\to 0}{\text{Lt}} \frac{\sin\theta}{\theta}=1.$

Let $\angle AOP=\theta$ radians.

With O as centre and any radius draw a circle cutting OP, OA in P and A. Draw PM $\perp$ OA and produce it to cut the circle in Q.



Draw the tangent at P and produce it to meet OA in T. Join TQ and OQ.

Then the right-angled triangles OMP and OMQ are congruent.

$\therefore$ MP$=$MQ and arc PA$=$arc QA.

Again, from the congruent triangles OPT and OQT, TP$=$TQ.

Now chord PQ < arc PQ < (PT + TQ)

$\therefore$ 2PM < 2 arc PA < 2PT

or PM < arc PA < PT

$$\therefore \quad \frac{MP}{OP} < \frac{\text{arc } PA}{OP} < \frac{PT}{OP}$$

i.e. $\sin\theta < \theta < \tan\theta \left[\because \quad \theta = \frac{l}{r} \text{ from trigonometry}\right]$

or $$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

or $$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

i.e. $\frac{\sin\theta}{\theta}$ lies between 1 and $\cos\theta$.

But $\cos\theta \to 1$ as $\theta \to 0$.

$\therefore$ $\frac{\sin\theta}{\theta}$ approaches 1 as θ approaches 0.

Hence $\underset{\theta\to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1$.

2. $\underset{\theta\to 0}{\text{Lt}} \frac{\tan\theta}{\theta} = 1$.

Sol. $$\text{L.H.S.} = \underset{\theta\to 0}{\text{Lt}} \frac{\tan\theta}{\theta}$$

$$= \underset{\theta\to 0}{\text{Lt}} \frac{\frac{\sin\theta}{\theta}}{\cos\theta}$$

$$= \frac{\underset{\theta\to 0}{\text{Lt}} \frac{\sin\theta}{\theta}}{\underset{\theta\to 0}{\text{Lt}} \cos\theta}$$

$$= \frac{1}{1} = 1.$$

3. $\underset{x\to a}{\text{Lt}} \frac{x^n - a^n}{x - a} = na^{n-1}$.

Solution. Let $x = a + h$ so that $h \to 0$ as $x \to a$

$$\therefore \quad \underset{x\to a}{\text{Lt}} \frac{x^n - a^n}{x - a} = \underset{h\to 0}{\text{Lt}} \frac{(a+h)^n - a^n}{a + h - a}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{a^n\left(1 + \frac{h}{a}\right)^n - a^n}{h} \quad \text{[Expanding by Binomial Theorem]}$$

$$= \underset{h\to 0}{\mathrm{Lt}} \frac{a^n\left[1+\frac{nh}{a}+\frac{n(n-1)}{2\,!}\left(\frac{h}{a}\right)^2+\ldots\ldots\right]-a^n}{h}$$

$$= \underset{h\to 0}{\mathrm{Lt}} \frac{a^n+na^{n-1}h+\frac{n(n-1)}{2\,!}a^{n-2}h^2+\ldots-a^n}{h}$$

$$= \underset{h\to 0}{\mathrm{Lt}} \left[na^{n-1}+\frac{n(n-1)}{2\,!}a^{n-2}h+\ldots\ldots\right]$$

$$=na^{n-1}.$$

Worked out Examples

Example 1. Evaluate $\underset{x\to 2}{\mathrm{Lt}} \dfrac{2x^2+x-10}{3x^2+x-14}$.

Sol. The expression takes $\dfrac{0}{0}$ form if we substitute in it 2 for x.

Therefore, we write

$$\underset{x\to 2}{\mathrm{Lt}} \frac{2x^2+x-10}{3x^2+x-14}=\underset{x\to 2}{\mathrm{Lt}} \frac{(x-2)(2x+5)}{(x-2)(3x+7)}$$

$$=\underset{x\to 2}{\mathrm{Lt}} \frac{2x+5}{3x+7}$$

$$=\frac{4+5}{6+7}=\frac{9}{13}.$$

Or thus,

Let $x=2+h$, so that as $x\to 2$, $h\to 0$.

$$\therefore \quad \underset{x\to 2}{\mathrm{Lt}} \frac{2x^2+x-10}{3x^2+x-14}=\underset{h\to 0}{\mathrm{Lt}} \frac{2(2+h)^2+(2+h)-10}{3(2+h)^2+(2+h)-14}$$

$$=\underset{h\to 0}{\mathrm{Lt}} \frac{2h^2+9h}{3h^2+13h}$$

$$=\underset{h\to 0}{\mathrm{Lt}} \frac{2h+9}{3h+13}=\frac{9}{13}.$$

Example 2. Evaluate $\underset{x\to \infty}{\mathrm{Lt}} \dfrac{5x^2+7x-3}{4x^2-2x+7}$.

Sol. $$\underset{x\to \infty}{\mathrm{Lt}} \frac{5x^2+7x-3}{4x^2-2x+7}=\underset{x\to \infty}{\mathrm{Lt}} \frac{5+\frac{7}{x}-\frac{3}{x^2}}{4-\frac{2}{x}+\frac{7}{x^2}}.$$

(Dividing the numerator and the denominator by x^2)

Now, as $x \to \infty$, $\frac{1}{x}$ and $\frac{1}{x^2} \to 0$.

Hence the required limit$=\frac{5}{4}$.

Or thus,

Let $x=\frac{1}{y}$, so that as $x \to \infty$, $y \to 0$

$$\therefore \quad \underset{x\to\infty}{\text{Lt}} \frac{5x^2+7x-3}{4x^2-2x+7} = \underset{y\to 0}{\text{Lt}} \frac{\frac{5}{y^2}+\frac{7}{y}-3}{\frac{4}{y^2}-\frac{2}{y}+7}$$

$$= \underset{y\to 0}{\text{Lt}} \frac{5+7y-3y^2}{4-2y+7y^2}$$

$$=\frac{5}{4}.$$

Example 3. Evaluate $\underset{x\to\infty}{\text{Lt}} [\sqrt{x^2+5x+4}-\sqrt{x^2-3x+4}]$.

Sol. $\underset{x\to\infty}{\text{Lt}} [\sqrt{x^2+5x+4}-\sqrt{x^2-3x+4}]$

$$= \underset{x\to\infty}{\text{Lt}} \frac{[\sqrt{x^2+5x+4}-\sqrt{x^2-3x+4}][\sqrt{x^2+5x+4}+\sqrt{x^2-3x+4}]}{\sqrt{x^2+5x+4}+\sqrt{x^2-3x+4}}$$

(Please note this step)

$$= \underset{x\to\infty}{\text{Lt}} \frac{(x^2+5x+4)-(x^2-3x+4)}{\sqrt{x^2+5x+4}+\sqrt{x^2-3x+4}}$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{8x}{\sqrt{x^2+5x+4}+\sqrt{x^2-3x+4}}$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{8}{\sqrt{1+\frac{5}{x}+\frac{4}{x^2}}+\sqrt{1-\frac{3}{x}+\frac{4}{x^2}}}$$

(Dividing the numerator and denominator by x)

$$\frac{8}{2}=4.$$

Example 4. Evaluate (*i*) $\underset{x\to 0}{\text{Lt}} \left(\frac{\sin ax}{\sin bx}\right)^k$.

(*ii*) $\underset{x\to 0}{\text{Lt}} \frac{1-\cos x}{x^2}$. (*iii*) $\underset{h\to 0}{\text{Lt}} \frac{\sin(x+h)-\sin x}{h}$.

Sol. (*i*) $\underset{x\to 0}{\text{Lt}} \left(\dfrac{\sin ax}{\sin bx}\right)^k = \underset{x\to 0}{\text{Lt}} \left(\dfrac{\dfrac{\sin ax}{ax}}{\dfrac{\sin bx}{bx}}\right)^k \times \dfrac{(ax)^k}{(bx)^k}$

(Please note this step)

Now $\underset{x\to 0}{\text{Lt}} \dfrac{\sin ax}{ax} = \underset{x\to 0}{\text{Lt}} \dfrac{\sin bx}{bx} = 1.$

$\therefore$ Required limit $= \left(\dfrac{a}{b}\right)^k.$

(*ii*) $\underset{x\to 0}{\text{Lt}} \dfrac{1-\cos x}{x^2} = \underset{x\to 0}{\text{Lt}} \dfrac{2\sin^2 \dfrac{x}{2}}{x^2}$

$$= \underset{x\to 0}{\text{Lt}} \frac{2\sin^2 \dfrac{x}{2}}{\dfrac{x^2}{4}} \times \frac{1}{4}$$

$$= \text{Lt} \left(\frac{\sin \dfrac{x}{2}}{\dfrac{x}{2}}\right)^2 \times \frac{1}{2}$$

Now $\underset{x\to 0}{\text{Lt}} \dfrac{\sin \dfrac{x}{2}}{\dfrac{x}{2}} = 1$

$\therefore$ The required limit $= \dfrac{1}{2}$

$\underset{x\to 0}{\text{Lt}} \dfrac{1-\cos x}{x^2} = \underset{x\to 0}{\text{Lt}} \dfrac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$

$$= \underset{x\to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^2 \times \frac{1}{1+\cos x} = \frac{1}{2}.$$

(*iii*) $\underset{h\to 0}{\text{Lt}} \dfrac{\sin (x+h) - \sin x}{h} = \underset{x\to 0}{\text{Lt}} \dfrac{2\cos\left(x+\dfrac{h}{2}\right)\sin \dfrac{h}{2}}{h}.$

$$= \underset{h\to 0}{\text{Lt}} \cos \frac{\left(x+\dfrac{h}{2}\right)\sin \dfrac{h}{2}}{\dfrac{h}{2}}.$$

(Please note this step)

Now $\underset{h\to 0}{\text{Lt}} \dfrac{\sin \frac{h}{2}}{\frac{h}{2}}=1$ and $\underset{h\to 0}{\text{Lt}} \cos\left(x+\dfrac{h}{2}\right)=\cos x$

$\therefore$ The required limit$=\cos x$.

1·11. Theorems on limits

(*The proofs of these theorems may be taken for granted at this stage*)

If $\underset{x\to a}{\text{Lt}}\ f(x)=l$ and $\underset{x\to a}{\text{Lt}}\ \phi(x)=m$, then

(*i*) $\underset{x\to a}{\text{Lt}}\ [f(x)\pm\phi(x)]=\underset{x\to a}{\text{Lt}}\ f(x)\pm\underset{x\to a}{\text{Lt}}\ \phi(x)$

$$=l\pm m.$$

(*ii*) $\underset{x\to a}{\text{Lt}}\ [f(x)\phi)]=\underset{x\to a}{\text{Lt}}\ f(x)\times\underset{x\to a}{\text{Lt}}\ \phi(x)=lm.$

(*iii*) $\underset{x\to a}{\text{Lt}}\ \dfrac{f(x)}{\phi(x)}=\dfrac{\underset{x\to a}{\text{Lt}}\ f(x)}{\underset{x\to a}{\text{Lt}}\ \phi(x)}=\dfrac{l}{m}$ provided $m\neq 0$.

(*iv*) $\underset{x\to a}{\text{Lt}}\ [cf(x)]=c\ \underset{x\to a}{\text{Lt}}\ f(x)=cl$, where c is a constant.

(*v*) $\underset{x\to a}{\text{Lt}}\ \log f(x)=\log\ \underset{x\to a}{\text{Lt}}\ f(x)=\log l$ provided $l>0$.

Exercise III

Evaluate the following :

1. $\underset{x\to 5}{\text{Lt}}\ \dfrac{x^2-25}{x-5}$.

2. $\underset{x\to 3}{\text{Lt}}\ \dfrac{3x^2-4x-15}{5x^2-9x-18}$.

3. $\underset{x\to a}{\text{Lt}}\ \dfrac{x^m-a^m}{x-a}$.

4. $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sqrt{1+x}-1}{x}$. (*K.U. 1976*)

5. $\underset{x\to\infty}{\text{Lt}}\ \dfrac{3x^2-4x-15}{5x^2-9x-18}$.

6. $\underset{x\to\infty}{\text{Lt}}\ \dfrac{3x^2+2x+1}{2x^2+3x+5}$.

7. $\underset{x\to\infty}{\text{Lt}}\ \dfrac{n(n+1)}{(n+2)(n+3)}$.

8. $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

9. $\underset{x\to 0}{\text{Lt}}\ \dfrac{x}{1-\sqrt{1-x}}$.

 [**Hint** : rationalize it.]

10. $\underset{x\to 0}{\text{Lt}}\ [\sqrt{x+1}-\sqrt{x}]$.

11. $\underset{n\to\infty}{\text{Lt}}\ \left(\dfrac{1}{3}+\dfrac{1}{3^2}+\dfrac{1}{3^3}+\ldots\ldots+\dfrac{1}{3^n}\right)$.

12. $\underset{x\to\infty}{\text{Lt}}\ (a+ar+ar^2+\ldots\ldots+ar^{n-1})$; $|r|<1$.

13. $\underset{x \to \infty}{\text{Lt}} \dfrac{1}{n^4}(1^3+2^3+3^3+\ldots\ldots+n^3)$.

14. $\underset{x \to 0}{\text{Lt}} \dfrac{\tan x - \sin x}{x^3}$.

15. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin mx}{\sin nx}$.

16. $\underset{x \to 0}{\text{Lt}} \dfrac{\tan (x+h) - \tan x}{h}$.

17. $\underset{x \to 0}{\text{Lt}} \dfrac{\text{cosec } x - \cot x}{x}$

18. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin 3x \cos 2x}{\sin 2x}$.

19. $\underset{\theta \to \pi/2}{\text{Lt}} \dfrac{\cot \theta - \cos \theta}{\cos^3 \theta}$. *(K.U. 1976)*

20. $\underset{\theta \to 0}{\text{Lt}} \dfrac{\tan \theta - \sin \theta}{\theta^3}$.

Show that :

21. $\underset{x \to a}{\text{Lt}} \dfrac{x^m - a^m}{x^n - a^n} = \dfrac{m}{n} a^{m-n}$ when $m > n$.

22. $\underset{x \to 0}{\text{Lt}} \dfrac{\cos ax - \cos bx}{x^2} = \dfrac{b^2 - a^2}{2}$.

23. $\underset{x \to 0}{\text{Lt}} \dfrac{\sqrt{1+x+x^2} - 1}{x} = \dfrac{1}{2}$.

24. $\underset{h \to 0}{\text{Lt}} \dfrac{\sec (x+h) - \sec x}{h} = \sec x \tan x$.

25. If $f(x) = x^3$, show that

$$\underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h} = 3x^2.$$

26. $\underset{x \to 1}{\text{Lt}} \dfrac{x^m - 1}{x^n - 1}$. *(K.U. 1975)*

27. $\underset{x \to \pi/2}{\text{Lt}} (\sec x - \tan x)$. *(K.U. 1975)*

28. $\underset{x \to \infty}{\text{Lt}} x \sin \dfrac{1}{x}$. *(K.U. 1975)*

29. $\underset{h \to 0}{\text{Lt}} \dfrac{\dfrac{1}{5+h} - \dfrac{1}{5}}{h}$. *(K.U. 1975)*

30. $\underset{x \to 0}{\text{Lt}} \dfrac{\sqrt{1+x^2} - \sqrt{1+x^2}}{x^2}$ *(K.U. 1975)*

2

Differentiation

2·1. If a variable x changes from one value x_1 to another value x_2, then the difference between the two values is called the *increment* in x, and is *usually* denoted by δx (read as "delta x").

2·2. Differential Coefficient

Def. The differential coefficient of a function is defined as follows :

Let $f(x)$ be any function of x and $f(x+\delta x)$ the same function of $x+\delta x$, then the *limit value* of the expression $\dfrac{f(x+\delta x)-f(x)}{\delta x}$ as δx approaches zero is called the *Differential Coefficient* (or the Derivative) of $f(x)$ with respect to x and is denoted by $f'(x)$.

2·3. Another form of the definition of the Differential Coefficient

Let $y=f(x)$ be a given function of x. If we give to x a small increment δx and the corresponding small increment δy to y, then the *limiting value* of the ratio $\dfrac{\delta y}{\delta x}$ as $\delta x \to 0$ is called the *Differential coefficient* (or the *Derivative*) of $f(x)$ or of y and is *generally* written as

$$\frac{d}{dx}[f(x)] \quad \text{or} \quad \frac{dy}{dx} \quad \text{or } f'(x), \text{ etc.}$$

Thus if $\qquad y=f(x)$

We have $\qquad y+\delta y=f(x+\delta x)$

$$\therefore \qquad \delta y=f(x+\delta x)-f(x)$$

$$\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\therefore \qquad \frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\delta y}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{f(x+\delta x)-f(x)}{\delta x}.$$

Note 1. $\dfrac{dy}{dx}$ is read as "dee-wy-over-dee-eks".

Note 2. In the ratio $\frac{\delta y}{\delta x}$, δx and δy being small increments in x and y respectively have meaning though taken separately as well, while dx and dy in $\frac{dy}{dx}$ have no meaning when taken separately. On the contrary, $\frac{dy}{dx}$ is a *single* quantity standing for $\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x}$.

2·4. Working Rule for finding the differential coefficient of a given function f(x)

1. Put $y=f(x)$.

2. Give to x a small increment δx, and the corresponding small increment δy to the dependent variable y.

Thus if $\qquad y=f(x) \qquad$ ...(*i*)

we have $\qquad y+\delta y=f(x+\delta x) \qquad$ ...(*ii*)

3. Subtract (*i*) from (*ii*), then

$$\delta y=f(x+\delta x)-f(x) \qquad \text{...}(iii)$$

4. Divide both sides of (*iii*) by δx.

5. Proceed to the limits when $\delta x \to 0$

and $\qquad$ denote $\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x}$ by $\frac{dy}{dx}$.

Note. 1. From (*iii*) it is easy to see that

$$\delta y \to 0 \text{ as } \delta x \to 0.$$

Note 2. Differentiating a given function in the manner explained in Article 2·4 as known as differentiating the given function *ab-initio* or *from first principles* or *from definition.*

2·5. Geometrical Significance of $\frac{dy}{dx}$

Let P(x, y) be any point on the graph of the function $y=f(x)$ and Q ($x+\delta x$, $y+\delta y$) be the neighbouring point on it.

Draw PL and QM perpendiculars on OX and PR perpendicular on QM. Let the secant PQ on being produced meet x-axis in N making an angle θ with its positive direction.

Now $\quad$ PR=LM=OM−OL=$(x+\delta x)-x=\delta x$

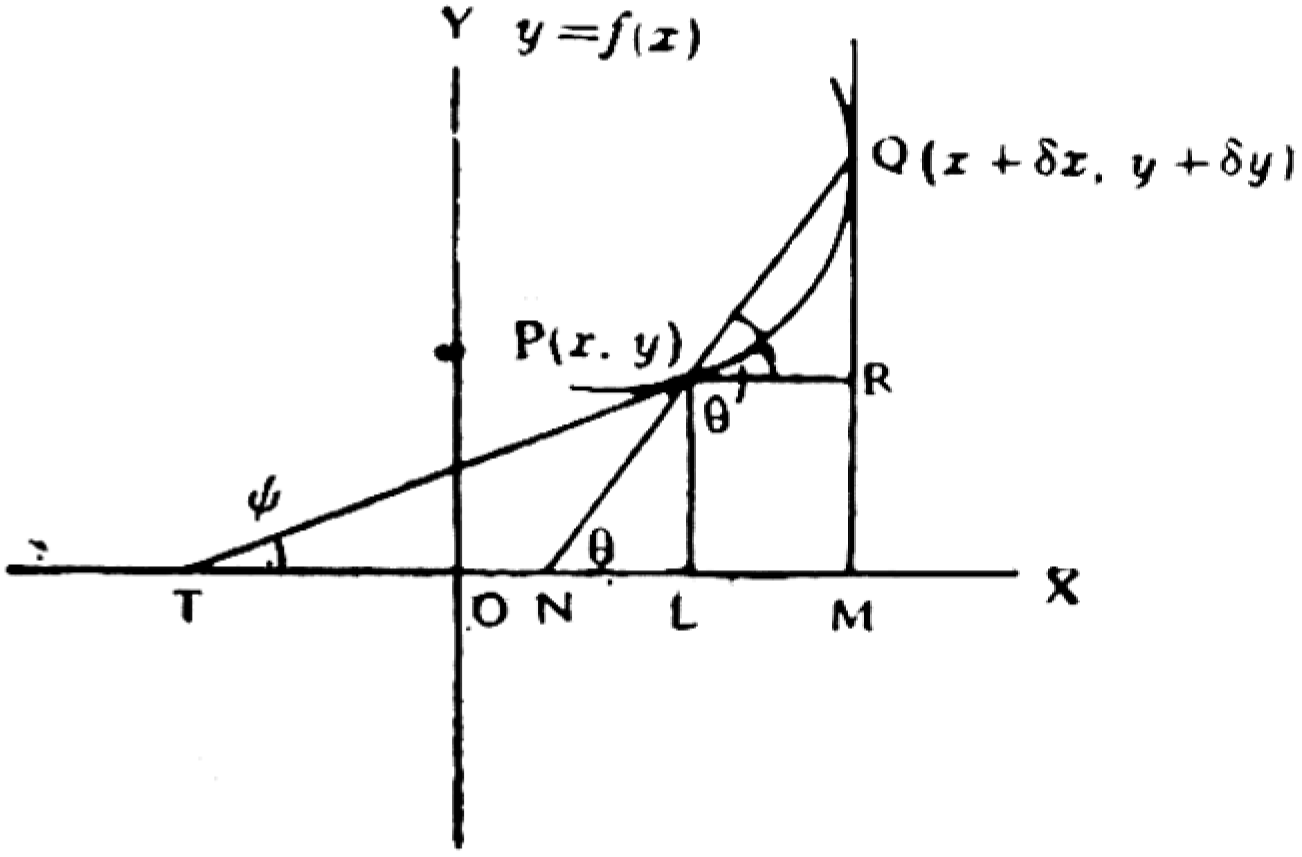
and $\quad$ QR=QM−MR=QM−PL=$(y+\delta y)-y=\delta y$.

$\therefore$ From the right $\angle$d $\triangle$PQR, we have

$$\frac{\delta y}{\delta x}=\tan\theta.$$

Now, as Q begins to move towards P along the curve, δx decreases, and when Q finally *coincides* with P, δx becomes infinitely

small and approaches zero. Consequently, the secant PQ becomes tangent to the curve at P and the angle θ is changed to angle ψ,



which this tangent makes with the positive direction of x-axis. This means that

$$\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{\delta y}{\delta x}=\underset{Q\to P}{\mathrm{Lt}}\ \tan\theta=\tan\psi.$$

But from the definition of differential coefficient (Article 2·3), we know that

$$\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{\delta y}{\delta x}=\frac{dy}{dx}$$

$$\therefore\qquad \frac{dy}{dx}=\tan\psi=\text{slope of the tangent.}$$

Hence, *the value of the differential coefficient of a function* $f(x)$ *at a point is the slope of the tangent to the graph of the function at that point.*

2·6. Differential Coefficient of

(*i*) $\mathbf{x^n}$ (*ii*) $\mathbf{(ax+b)^n}$.

(*i*) Let $y=x^n$

Then $y+\delta y=(x+\delta x)^n$

$\therefore$ $\delta y=(x+\delta x)^n-y$

$=(x+\delta x)^n-x^n$

Now $\delta y=x^n\left[\left(1+\dfrac{\delta x}{x}\right)^n-1\right]$

$=x^n\left[\left\{1+n.\dfrac{\delta x}{x}+\dfrac{n(n-1)}{21}\left(\dfrac{\delta x}{x}\right)^2+\text{terms}\right.\right.$ involving third and higher powers of $\left.\left.\delta x\right\}-1\right]$

$$=x^n\left[n\frac{\delta x}{x}+\frac{n(n-1)}{2\,!}\left(\frac{\delta x}{x}\right)^2+\ldots\ldots\right]$$

$$\therefore\quad \frac{\delta y}{\delta x}=\frac{x^n\left[n\frac{\delta x}{x}+\frac{n(n-1)}{2\,!}\left(\frac{\delta x}{x}\right)^2+\ldots\ldots\right]}{\delta x}$$

$$=nx^{n-1}+\frac{n(n-1)x^{n-2}}{2\,!}\delta x+\text{ terms involving}$$

second and higher powers of δx.

Hence,
$$\frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta y}{\delta x}$$

$$=\underset{\delta x\to 0}{\mathrm{Lt}}\left[nx^{n-1}+\frac{n(n-1)x^{n-2}}{2\,!}.\delta x+\ldots\ldots\right]$$

$$=nx^{n-1}.$$

(*ii*) Let
$$y=(ax+b)^n \qquad \ldots(1)$$

$$y+\delta y=[a(x+\delta x)+b]^n$$

$$=[ax+b+a\delta x]^n$$

$$=\left[(ax+b)\left(1+\frac{a\delta x}{ax+b}\right)\right]^n$$

$$=(ax+b)^n\left[1+\frac{na\delta x}{(ax+b)}+\frac{n(n-1)}{2\,!}\cdot\frac{a^2}{(ax+b)^2}(\delta x)^2+\ldots\ldots\right]$$

$$=(ax+b)^n+n(ax+b)^{n-1}a\delta x+\frac{n(n-1)}{2\,!}(ax+b)^{n-2}\,a^2(\delta x)^2+\ldots \qquad \ldots(2)$$

Subtracting (1) from (2), we have

$$\delta y=n(ax+b)^{n-1}a\ \delta x+\frac{n(n-1)}{2\,!}\ (ax+b)^{n-2}a^2(\delta x)^2+\ldots\ldots$$

Dividing by δx,

$$\frac{\delta y}{\delta x}=n\ (ax+b)^{n-1}.a+\frac{n(n-1)}{2\,!}(ax+b)^{n-2}a^2.\delta x+\ldots\ldots$$

$$\therefore\quad \frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{dy}{dx}$$

$$=\underset{\delta x\to 0}{\mathrm{Lt}}\left[n(ax+b)^{n-1}\,a+\frac{n(n-1)}{2\,!}(ax+b)^{n-2}.a^2.\delta x\right.$$
$$\left.+\text{terms involving second and higher powers of }\delta x\right]$$

$$=na(ax+b)^{n-1}.$$

2·7. Fundamental Theorems of Differentiation

Theorem I. *The differential coefficient of a constant is zero.*

Proof. Let $\quad y=c,$ where c is a constant

$$y+\delta y=c \qquad \text{(No increment is given to a constant)}$$

$$\therefore \qquad \delta y=c-y=c-c=0$$

or $$\frac{\delta y}{\delta x}=0.$$

Hence $$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{dy}{dx}=0.$$

Theorem II. *An additive constant disappears in differentiation.*

Proof. Let $\quad y=f(x)+c$

$$y+\delta y=f(x+\delta x)+c$$

$$\delta y=f(x+\delta x)-f(x)$$

$$\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\therefore \qquad \frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}=\underset{\delta x\to 0}{\text{Lt}}\frac{f(x+\delta x)-f(x)}{\delta x}$$

Thus "c" has diappeared from the process.

Theorem III. *The differential coefficient of the product of a constant and a function is equal to the product of the constant and the differential coefficient of the function.*

Proof. Let $\quad y=cf(x)$

$$y+\delta y=cf(x+\delta x)$$

$$\delta y=cf(x+\delta x)-cf(x)$$

$$=c[f(x+\delta x)-f(x)]$$

$$\therefore \qquad \frac{\delta y}{\delta x}=c\,\frac{f(x+\delta x)-f(x)}{\delta x}$$

Hence $$\frac{\delta y}{\delta x}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}=c\underset{\delta x\to 0}{\text{Lt}}\frac{f(x+\delta x)-f(x)}{\delta x}.$$

Theorem IV. *The differential coefficient of the algebraic sum of several functions is equal to the algebraic sum of their differential coefficients.*

Proof. Let $\quad y=u+v-w+\ldots\ldots \qquad \ldots(1)$

when $u, v, w, \ldots\ldots$ are functions of x and y is also a function of x.

Given an increment δx to x, then as $u, v, w, \ldots\ldots, y$ are functions of x, the reactive increments $\delta u, \delta v, \delta w, \ldots\ldots, \delta y$ respectively.

Then $\quad y+\delta y=(u+\delta u)+(v+\delta v)-(w+\delta w)+\ldots\ldots \qquad \ldots(2)$

Subtracting (1) from (2), we have

$$\delta y=\delta u+\delta v-\delta w+\ldots\ldots$$

or
$$\frac{\delta y}{\delta x}=\frac{\delta u}{\delta x}+\frac{\delta v}{\delta x}-\frac{\delta w}{\delta x}+\ldots\ldots$$

$$\therefore \quad \frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta u}{\delta x}+\underset{\delta x\to 0}{\text{Lt}}\frac{\delta v}{\delta x}-\underset{\delta x\to 0}{\text{Lt}}\frac{\delta w}{\delta x}+\ldots\ldots$$

$$=\frac{du}{dx}+\frac{dv}{dx}-\frac{dw}{dx}+\ldots\ldots$$

Hence $\dfrac{d}{dx}(u+v-w+\ldots\ldots)=\dfrac{du}{dx}+\dfrac{dv}{dx}-\dfrac{dw}{dx}+\ldots\ldots$

Solved Examples

Example 1. Find *ab-initio*, the differential coefficient of

(*i*) $\sqrt{x}$. (*K.U. 1975*) (*ii*) $\dfrac{1}{x+a}$.

Sol. Let $y=\sqrt{x}$

then
$$y+\delta y=\sqrt{x+\delta x}$$

$$\delta y=\sqrt{x+\delta x}-\sqrt{x}$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}$$

$$=\frac{\delta x}{\delta x(\sqrt{x+\delta x}+\sqrt{x})}$$

(Please note this step)

$$=\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$$

Hence
$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}.$$

(*ii*) Let
$$y=\frac{1}{x+a}$$

then
$$y+\delta y=\frac{1}{(x+\delta x)+a}$$

$$\delta y=\frac{1}{(x+\delta x)+a}-\frac{1}{x+a}$$

$$=\frac{(x+a)-[(x+\delta x)+a]}{(x+a)[(x+\delta x)+a]}$$

$$=\frac{-\delta x}{(x+a)[(x+\delta x)+a]}$$

$$\therefore \qquad \frac{\delta y}{dx}=-\frac{1}{(x+a)[(x+\delta x)+a]}$$

Hence $$\frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta y}{\delta x}=-\frac{1}{(x+a)^2}.$$

Example 2. If $y=Ax^3+Bx^2+Cx+D$, prove from first principles $\frac{dy}{dx}=3Ax^2+2Bx+C$.

Sol. $$y=Ax^3+Bx^2+Cx+D$$

then $$y+\delta y=A(x+\delta x)^3+B(x+\delta x)^2+C(x+\delta x)+D$$

or $$y+\delta y=Ax^3+3Ax^2\delta x+3Ax(\delta x)^2+A(\delta x)^3 +Bx^2+2Bx.\delta x+B(\delta x)^2+Cx+C\delta x+D$$

or $$\delta y=3Ax^2\delta x+3Ax(\delta x)^2+A(\delta x)^3+2Bx.\delta x +B(\delta x)^2+C\delta x$$

$$\therefore \qquad \frac{\delta y}{\delta x}=3Ax^2+3Ax.\delta x+A(\delta x)^2+2Bx+B(\delta x)+C$$

Hence $$\frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta y}{\delta x}=\underset{\delta x\to 0}{\mathrm{Lt}}[3Ax^2+3Ax.\delta x+A(\delta x)^2+2Bx+B(\delta x)+C]$$

$$=3Ax^2+2Bx+C.$$

Exercise IV

1. Write down the derivatives of the following :

(*i*) x^5. (*ii*) $\frac{1}{x}$. (*iii*) $\sqrt{x}$. (*iv*) x^3. (*v*) $\frac{1}{x^2}$.

(*vi*) $(2x+3)^{10}$. (*vii*) $(8-13x)^{2/3}$. (*viii*) $\frac{1}{\sqrt{5x+7}}$.

2. Find the differential coefficients of :

(*i*) $x(1-x)$. (*ii*) $x^3-6x^2+11x-6$.

(*iii*) $x^4+5x^3-7x^2+4x+5$. (*iv*) $\frac{1}{x^2}+c$.

(*v*) $\frac{x^4+3x^3+4}{x^2}$. (*vi*) $mx+\frac{a}{m}$.

(*vii*) $7x^{-4}+8x^{-7}$.

3. (*i*) If $\phi(x)=px^m+qx^n$, find $\phi'(x)$.

(*ii*) If $f(x)=x^4+4x^3+6x^2+4x+1$, find $f'(x)$.

(*iii*) If $y=1-2t+3t^2-4t^3$, find $\frac{dy}{dt}$.

(*iv*) Find $\dfrac{dy}{dx}$ at $x=2$, if $y=x^3+3x^2+3x+1$.

(*v*) If $y=x^2+5x+6$, find $\dfrac{dy}{dx}$ at $x=3$.

4. Find *ab-initio*, the derivatives of :

(*i*) $5x^3$. (*ii*) $(3x+4)^2$. (*iii*) $\dfrac{1}{\sqrt{x}}$.

(*iv*) ax^2+bx+c. (*v*) $\dfrac{1}{3x+1}$.

(*vi*) $\sqrt[3]{x}$. (*vii*) $\dfrac{3x+1}{5x+7}$. (*viii*) $\sqrt{ax+b}$.

(*ix*) $\dfrac{x+1}{\sqrt{x}}$. (*x*) $\dfrac{1}{\sqrt{ax+b}}$. (*xi*) $\dfrac{ax+b}{cx+d}$.

(*xii*) $\dfrac{x^3+3x^2-18}{x+3}$. (*xiii*) $\dfrac{1}{\sqrt{x+c}}$. *(K.U. 1977)*

5. If A denotes the area of a circle whose radius is r, show that $\dfrac{dA}{dr}$ is equal to the circumference of the circle.

[**Hint.** Area$=\pi r^2$].

6. If v is the volume of a sphere with radius r, show that $\dfrac{dv}{dr}$ is equal to the circumference of the sphere.

2·8. Differential coefficient of the product of two functions

Theorem V. *The differential coefficient of the product of two functions is equal to the first function $\times$ the differential coefficient of the second function $+$ second function $\times$ the differential coefficient of the first function.* Or symbolically

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}, \text{ where } u \text{ and } v \text{ are both functions of } x.$$

Proof. Let $y=uv$ and let δx, δu, δv and δy be the increments of x, u, v and y respectively, then

$$y+\delta y=(u+\delta u)(v+\delta v)$$

$$\therefore \quad \delta y=(u+\delta u)(v+\delta v)-uv$$

$$=uv+u.\delta v+v.\delta u+\delta u.\delta v-uv$$

$$=u\delta v+(v+\delta v)\delta u$$

or

$$\frac{\delta y}{\delta x}=u\frac{\delta v}{\delta x}+(v+\delta v)\frac{\delta u}{\delta x}.$$

Proceeding to the limits as $\delta x\to 0$ and consequently δu, δv, $\delta y\to 0$.

Hence, $\dfrac{dy}{dx}=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$.

Cor. If $y=uvw$, where u, v, w are all functions of x, then

$$\frac{dy}{dx}=uv\frac{dw}{dx}+vw\frac{du}{dx}+uw\frac{dv}{dx}.$$

Proof. Let $z=uv$, so that

$$y=zw$$

$\therefore$ By the above theorem, we have

$$\frac{dy}{dx}=z\frac{dw}{dx}+w\frac{dz}{dx} \qquad \ldots(1)$$

and $$\frac{dz}{dx}=u\frac{dv}{dx}=v\frac{du}{dx} \qquad \ldots(2)$$

Substituting the value of $\dfrac{dz}{dx}$ from (2) in (1), we get

$$\frac{dy}{dx}=z\frac{dw}{dx}+w\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$=uv\frac{dw}{dx}+uw\frac{dv}{dx}+vw\frac{du}{dx}$$

$$y=uvw \qquad \ldots(i)$$

$\therefore$ $$y+\delta y=(u+\delta u)(v+\delta v)(w+\delta w) \qquad \ldots(ii)$$

Substituting, we have

$$\delta y=(u+\delta u)(v+\delta v)(w+\delta w)-uvw$$

$$\delta y=uvw+vw\delta u+uw\delta v+uv\delta w+u\delta v\delta w+v\delta w\delta u$$
$$+w\delta u\delta v+\delta u\delta v\delta w-uvw$$

$\therefore$ $$\frac{\delta y}{\delta x}=vw\frac{\delta u}{\delta x}+uw\frac{\delta v}{\delta x}+uv\frac{\delta w}{\delta x}+u\frac{\delta v}{\delta x}.\delta w$$
$$+v\frac{\delta w}{\delta x}.\delta u+w\frac{\delta u}{\delta x}.\delta v+\frac{\delta u}{\delta x}.\delta v.\delta w$$

Let $\delta x\to 0$,

$\therefore$ δu, δv, δw, δy all$\to 0$

$$\frac{dy}{dx}=vw\frac{du}{dx}+wu\frac{dv}{dx}+uv\frac{dw}{dx}.$$

2·9. The differential coefficient of the quotient of two functions.

Theorem VI. *The Differential Co-efficient of the quotient of two functions.*

$$=\frac{(\textit{Deno.})(\textit{Diff. Co-effi. of Numer.})-(\textit{Numer.})(\textit{Diff. Co-effi. of Deno.})}{\textit{Square of Denominator}}$$

or *symbolically* $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$ when u and v are functions of x.

Proof. Let $y=\dfrac{u}{v}$ and δx, δu and δv be the increments of x, u and v respectively, and δy the increment of y

then $$y+\delta y=\frac{u+\delta u}{v+\delta v}$$

$$\delta y=\frac{u+\delta u}{v+\delta v}-\frac{u}{v}$$

$$=\frac{uv+v.\delta u-uv-u.\delta v}{v(v+\delta v)}$$

$$=\frac{v\delta u-u.\delta v}{v(v+\delta v)}$$

$$\therefore \qquad \frac{\delta y}{\delta x}=\frac{v\dfrac{\delta u}{\delta x}-u\dfrac{\delta v}{\delta x}}{v(v+\delta v)}$$

Now as $\delta x\to 0$, $\delta v\to 0$, $\therefore \quad (v+\delta v)\to v$

Hence $$\frac{dy}{dx}=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}.$$

Solved Examples

Example 1. If $y=(2x+3)(4x+5)$, find $\dfrac{dy}{dx}$.

Sol. Here $y=(2x+3)(4x+5)$

$$\therefore \qquad \frac{dy}{dx}=(2x+3)\ \frac{d}{dx}\ (4x+5)+(4x+5)\frac{d}{dx}\ (2x+3)$$

$$=(2x+3)\times 4+(4x+5)\times 2$$

$$=16x+22.$$

Example 2. Find $\dfrac{dy}{dx}$ if $y=(x^2+1)(3x^3+9)(2x+4)$.

Sol. $$\frac{dy}{dx}=(x^2+1)(3x^3+9)\ \frac{d}{dx}\ (2x+4)+(x^2+1)(2x+4)$$

$$\frac{d}{dx}\ (3x^3+9)+(3x^3+9)(2x+4)\times\frac{d}{dx}\ (x^2+1)$$

$$=(x^2+1)(3x^3+9)\times 2+(x^2+1)(2x+4)\times 9x^2$$
$$+(3x^3+9)(2x+4)\times 2x$$

$$=2(x^2+1)(3x^3+9)+9x^2(x^2+1)(2x+4)+2x(3x^3+9)$$
$$\times(2x+4).$$

Example 3. If $y=\frac{x^2-1}{x^2+1}$, find $\frac{dy}{dx}$.

Sol. $$\frac{dy}{dx}=\frac{(x^2+1)\frac{d}{dx}(x^2-1)-(x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$
$$=\frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}$$
$$=\frac{4x}{(x^2+1)^2}.$$

Example 4. Use $\frac{u}{v}$ form to show that

$$\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{x^2}.$$

Sol. Here $u=1$ and $v=x$

$$\therefore \quad \frac{d}{dx}\left(\frac{1}{x}\right)=\frac{x\times 0-1.1}{x^2} \qquad \left[\because \frac{d}{dx}(1)=0\right]$$
$$=-\frac{1}{x^2}.$$

Note. *Questions on uv and $\frac{u}{v}$ forms from the forthcoming exercise should not be attempted till the following theorem is fully followed.*

2·10. Function of a Function

Sometimes it so happened that y is not defined directly as a function of x. On the contrary, a new variable, z (say) is introduced, and y is defined as a function of z, which, in turn, is defined as a function of x. This means that y is a function of z while z is itself a function of x. In such a case, *y is said to be a* **function of a function.**

For Example, if $$y=u^n \qquad \ldots(i)$$
and $$u=(x^3+3x^2+5x+1) \qquad \ldots(ii)$$
we say that y is a function of a function. If we eliminate u between (i) and (ii), we can get a direct relation between y and x, *viz.*, $y=(x^3+3x^2+5x+1)^n$ from this relation, we can get $\frac{dy}{dx}$ direct. But, sometimes this sort of elimination becomes very difficult. In such a case, we make use of the following theorem for finding $\frac{dy}{dx}$.

Theorem VII. *Given (i) $y=f(z)$ and*
(ii) $z=\phi(x)$,

to prove that

$$\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}.$$

Proof. Let δx, δz and δy be the increments of x, z and y respectively, such that

(*a*) z becomes $z+\delta z$ as y becomes $y+\delta y$ in (*i*), and

(*b*) x becomes $x+\delta x$, as z becomes $z+\delta z$ in (*ii*).

Now, by ordinary Algebra, $\frac{\delta y}{\delta x}=\frac{\delta y}{\delta z}\times\frac{\delta z}{\delta x}$, however small δx, δy or δz may be. Thus, in the limit, we have

$$\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta y}{\delta x}=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta y}{\delta z}\times\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta z}{\delta x}$$

or

$$\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}.$$

Cor. *Differential co-efficient of u^n, where u is a function of x,*

n^n is a function of u which, in turn, is a function of x.

$$\therefore\qquad \frac{d}{dx}(u^n)=\frac{d}{du}(u^n)\times\frac{du}{dx}=nu^{n-1}\frac{du}{dx}.$$

Note. The differential coefficient of u^n with respect to u is nu^{n-1}, while that with respect to x is $nu^{n-1}\frac{du}{dx}$. Similarly, the differential coefficient of x^n w.r.t. x is nx^{n-1}, whereas that w.r.t. y is $nx^{n-1}\frac{dx}{dy}$.

Thus, *it is very important to know with respect to what we are required to differentiate a certain function. When nothing is mentioned in this behalf, we can presume that the function is to be differentiated w.r.t. its argument.*

This means that if we are required to differentiate x^n, we have to differentiate it with respect to its argument which is x. Its differential co-efficient will, therefore, be nx^{n-1}. But, if we are required to differentiate x^n with respect to t, the differential co-efficient will be $nx^{n-1}\frac{dx}{dt}$.

Extension. The result of Theorem VII can be extended. If y is a function of u, u a function of v, and v a function of x, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}.$$

Theorem VIII. *Relation between* $\frac{dy}{dx}$ *and* $\frac{dx}{dy}$.

If y is a function of x, then we can express x in terms of y. Hence x can be regarded as a function of y. From the former relation we can calculate $\frac{dy}{dx}$ and from the latter $\frac{dx}{dy}$.

They are connected by the relation

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

Proof. By ordinary Algebra, we have

$$\frac{\delta y}{\delta x} \cdot \frac{\delta x}{\delta y} = 1$$

When $\delta x \to 0$, $\delta y \to 0$

$\therefore$ Taking limits, we get

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

Cor. $\frac{dy}{dx}$ and $\frac{dx}{dy}$ *are reciprocals of each other.*

Solved Examples

Example 1. Find $\frac{dy}{dx}$ if $y=(x^4+7x^2+3)^8$.

Sol. Let $\qquad u=x^4+7x^2+3 \qquad \ldots(i)$

so that $\qquad y=u^8. \qquad \ldots(ii)$

Now from (i), $\qquad \frac{du}{dx}=4x^3+14x$

and from (ii), $\qquad \frac{dy}{du}=8u^7.$

Hence
$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=8u^7\ (4x^3+14x)$$
$$=8(x^4+7x^2+3)^7\ (4x^3+14x)$$
$$=16x(2x^2+7)(x^4+7x^2+3)^7.$$

Alternative Method. The given expression $(x^4+7x^2+3)^8$ is a function of x^4+7x^2+3 which is, in turn, a function of x.

$\therefore \quad \frac{d}{dx}(x^4+7x^2+3)^8$

$$=8(x^4+7x^2+3)^7\times\frac{d}{dx}(x^4+7x^2+3)$$
$$=8(x^4+7x^2+3)(4x^3+14x)$$
$$=16x(2x^2+7)(x^4+7x^2+3).$$

Note. The student is advised to *completely* master the *alternative* method, as he can save a lot of time by making frequent use of it.

Example 2. Show by the method of the *function of a function* that the differential coefficient of $(ax+b)^n$ is $na(ax+b)^{n-1}$.

Sol. Let $y=(ax+b)^n$ and

$$u=(ax+b) \quad \ldots(i)$$

so that $y=u^n \quad \ldots(ii)$

From (i) $\dfrac{du}{dx}=a$ and from

(ii) $\dfrac{dy}{du}=nu^{n-1}$

Thus $$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=nu^{n-1}\times a=an(ax+b)^{n-1}$$

(After replacing u)

Alternatively

$$\frac{dy}{dx}=n(ax+b)^{n-1}\times\frac{d}{dx}(ax+b)$$

$$=n(ax+b)^{n-1}\times a$$

$$=an(ax+b)^{n-1}.$$

Example 3. Differentiate $\dfrac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}}$.

Sol. Let $$y=\frac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}} \quad \left(\frac{u}{v} \text{ form}\right)$$

Rationalizing the denominator, we get

$$y=\frac{[\sqrt{(x+a)}-\sqrt{x-a)}]^2}{(x+a)+(x-a)}$$

$$=\frac{x+a+x-a-2\sqrt{x^2-a^2}}{2a}$$

$$=\frac{x-\sqrt{x^2-a^2}}{a}$$

$$=\frac{1}{a}\left\{x-(x^2-a^2)^{\frac{1}{2}}\right\}$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{a}\left\{1-\frac{1}{2}(x^2-a^2)^{-\frac{1}{2}}\times\frac{d}{dx}(x^2-a^2)\right\}$$

$$=\frac{1}{a}\left\{1-\frac{1}{2}(x^2-a^2)^{-\frac{1}{2}}\times 2x\right\}$$

$$=\frac{1}{a}\left\{1-\frac{x}{\sqrt{x^2-a^2}}\right\}.$$

Note. In example (3), $\sqrt{x^2-a^2}$ has been differentiated by the method of the *function of a function*, and there too we have used the *short cut* method. This should be noted by the student in particular.

Exercise V

Differentiate the following with respect to x :

1. $(3x^2+7x+5)^8$.
2. $(ax^2+bx+c)^n$.
3. $\sqrt{2-x^6}$.
4. $(x^7+4x^3+21x)^{-6}$.
5. $\dfrac{8}{\sqrt{8x^2-7x+9}}$.
6. $(px^3-qx^2+rx+s)^{m/n}$.
7. $q\sqrt{(x^4+3x^2-1)^p}$.
8. $\sqrt{a^2-x^2}+\dfrac{1}{\sqrt{a^2+x^2}}$.
9. $\dfrac{1}{\sqrt[6]{7x^2-3}}$.
10. $[c^3+(c^2+x^2)^2]^5$.

Differentiate the following : (*u.v form*)

11. $(2x^2-3)(3x^3-2)$.
12. $\left(x+\dfrac{1}{x}\right)\left(2x-\dfrac{3}{x}\right)$.
13. $(x^2+7)(x^3+10)$.
14. $(x^p+q)(x^m+n)$.
15. $(x-1)^2(4x+7)$.
16. $\left(x+\dfrac{1}{x}\right)\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)$.
17. $\sqrt[3]{x^2(x+3)}$.
18. $(3x^2+4)^2(1-x^2)^3$.

Differentiate the following : $\left(\dfrac{u}{v}\ \textit{form}\right)$

19. $\dfrac{bx^5+c}{x^2+a}$.
20. $\sqrt{\dfrac{1+x}{1-x}}$. (*K.U. 1977*)
21. $\sqrt{\dfrac{x^2-2ax}{bx-cx^2}}$. (*P.U. 1952*)
22. $\dfrac{2(x-1)}{x^2+2x-3}$
23. $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$.
24. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}$.

2·11. Implicit and Explicit Functions

Def. "If the relation between x and y be solved for y in terms of x in the form of the simple equation

$$y=f(x)$$

then y is said to be an *Explicit* function of x. If the relation between x and y is not expressed in this form, then y is said to be an *Implicit* function of x."

2·12. Differentiation of y^n with respect to x

Or

Symbolically, $\dfrac{d}{dx}\left(y^n\right)$.

Let $u=y^n \ldots (i)$, then $\dfrac{d}{dx}\left(y^n\right)=\dfrac{du}{dx}$.

Now, by the *Function of a Function method*

$$\frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}$$

and from (i), $\dfrac{du}{dy}=ny^{n-1}$

Hence $\dfrac{du}{dx}=\dfrac{d}{dx}\left(y^n\right)\cdot\dfrac{dy}{dx}$

$$=ny^{n-1}\frac{dy}{dx}.$$

The student is advised to note this carefully. The differential coefficient of y^n with respect to x, as shown above, is $ny^{n-1}\dfrac{dy}{dx}$ and not only ny^{n-1}. On the other hand, ny^{n-1} is the differential coefficient of y^n when the latter is differentiated with respect to y.

Example. $3y^2\dfrac{dy}{dx}$ is the differential coefficient of y^3 with respect to x, whereas $3y^2$ is the differential coefficient of y^2 with respect to y itself.

2·13. Differentiator of an implicit function

When y is an implicit function of x, it is mixed up with x in such a way that at times it is difficult for us to express y in terms of x or x in terms of y. Consequently, we cannot easily find $\dfrac{dy}{dx}$ or $\dfrac{dx}{dy}$. In such a case, the following rule is laid down :

(1) Differentiate both sides of the equation with respect to x.

(2) Solve the resulting equation for $\dfrac{dy}{dx}$.

Example. If $ax^2+2hxy+by^2=1$, find $\dfrac{dy}{dx}$.

Sol. Differentiating both sides of the equation w.r.t. x, we have

$$\frac{d}{dx}(ax^2)+\frac{d}{dx}(2hxy)+\frac{d}{dx}(by^2)=\frac{d}{dx}(5)$$

or $$a\frac{d}{dx}(x^2)+2h\frac{d}{dx}(xy)+b\frac{d}{dx}(y^2)=\frac{d}{dx}(5)$$

or $$a\frac{d}{dx}(x^2)+2h\left\{y\frac{d}{dx}(x)+x\frac{d}{dx}(y)\right\}+b\frac{d}{dx}(y^2)=\frac{d}{dx}(5)$$

(Please note this step)

or $$a\times 2x+2h\left\{y\times 1+x\times 1\times\frac{dy}{dx}\right\}+2y\frac{dy}{dx}\times b=0$$

or $$ax+h\left(y+x\frac{dy}{dx}\right)+by\frac{dy}{dx}=0$$

or $$(hx+by)\frac{dy}{dx}=-(ax+hy)$$

Hence $$\frac{dy}{dx}=-\frac{ax+hy}{hx+by}$$

or $$ax^2+2hxy+by^2=1 \qquad \ldots(1)$$

giving increments

$$a(x+\delta x)^2+2h(x+\delta x)(y+\delta y)+b(y+\delta y)^2=1 \qquad \ldots(2)$$

Subtracting (2) from (1), we have

$$a(2x\delta x+\delta x^2)+2h(y\delta x+x\delta y+\delta x\delta y)+b(2y\delta y+\delta y^2)=0$$

Dividing by δx

$$a(2x+\delta x)+2h\left[y+x\frac{\delta y}{\delta x}+\delta y\right]+b\left[\frac{2y\delta y}{\delta x}+\frac{\delta y}{\delta x}.\delta y\right]=0$$

Let $\delta x\to 0$,

$\therefore$ δy also $\to 0$

$$\therefore\ a(2x)+2h\left[y+x\frac{dy}{dx}\right]+2by\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=-\frac{(ax+hy)}{(hx+by)}.$$

2·14. Differentiation of functions expressed in parametric forms

Sometimes y is not expressed as a function of x. On the other hand, both x and y are expressed as the functions of a third variable.

For instance, these are expressed as the functions of "t" by means of two different equations as shown below :

$$x=f(t)\ ;\ y=\phi(t).$$

In such a case, we have to eliminate "t" from the two equations before differentiating y w.r.t. x. But it is not always easy to effect such elimination, and obtain a direct relation between x and y. In such a case, $\dfrac{dy}{dx}$ is obtained by the *Function of Function Method.*

From the equation $x=f(t)$

we get $\dfrac{dx}{dt}$ as $f'(t)$, and from the equation

$$y=\phi(t)$$

we get $\dfrac{dy}{dt}$ as $\phi'(t)$. Finally $\dfrac{dy}{dx}$ is obtained as under :

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{f'(t)}{\phi'(t)}.$$

The method is illustrated below :

Example. $x=at^2$, $y=2at$, find $\dfrac{dy}{dx}$.

Sol. From $x=at^2$, $\dfrac{dx}{dt}=2at$

and from $y=2at$, $\dfrac{dy}{dt}=2a$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=2a\times\frac{1}{2at}=\frac{1}{t}.$$

Solved Examples

Example 1. If $x^m+y^m=a^m$, find $\dfrac{dy}{dx}$.

Sol. Differentiating both sides w.r.t. x, we have

$$mx^{m-1}+my^{m-1}\frac{dy}{dx}=0.$$

$$\therefore \qquad \frac{dy}{dx}=-\frac{x^{m-1}}{y^{m-1}}.$$

Example 2. Find $\dfrac{dy}{dx}$ when :

$$x=\frac{2t}{1+t^2},\quad y=\frac{1-t^2}{1+t^2}.$$

Sol. $x=\dfrac{2t}{1+t^2}$ | $y=\dfrac{1-t^2}{1+t^2}$

$\therefore \quad \dfrac{dx}{dt}=\dfrac{2(1+t^2)-2t\times 2t}{(1+t^2)^2}$ | $\therefore \quad \dfrac{dy}{dt}=\dfrac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2}$

$=\dfrac{2(1-t^2)}{(1+t^2)^2}$ | $=\dfrac{-4t}{(1-t^2)^2}$

Hence $\dfrac{dy}{dx}=\dfrac{dy}{dt}\times\dfrac{dt}{dx}=\dfrac{-4t}{(1+t^2)^2}\times\dfrac{(1+t^2)^2}{2(1-t^2)^2}=\dfrac{2t}{t^2-1}.$

Example 3. Differentiate $\dfrac{x^2}{1+x^2}$ with respect to x^2.

Sol. Here we have to find

$$\frac{d}{dx^2}\left(\frac{x^2}{1+x^2}\right).$$

$\therefore$ If $y=\dfrac{x^2}{1+x^2}$ and $z=x^2$, then we have to find $\dfrac{dy}{dz}$.

Now $y=\dfrac{x^2}{1+x^2}$ | $z=x^2$

$\therefore \quad \dfrac{dy}{dx}=\dfrac{2x(1+x^2)-2x\times x^2}{(1+x^2)^2}$ | $\therefore \quad \dfrac{dz}{dx}=2x \quad \ldots(2)$

$=\dfrac{2x}{(1+x^2)^2} \quad \ldots(1)$

$\therefore$ from (1) and (2), we get

$$\frac{dy}{dz}=\frac{dy}{dx}\times\frac{dx}{dz}=\frac{2x}{(1+x^2)^2}\times\frac{1}{2x}$$

$$=\frac{1}{(1+x^2)^2}.$$

Exercise VI

Find $\dfrac{dy}{dx}$ when :

1. $x=\dfrac{2t}{1+t^2}$, $y=\dfrac{1-t^2}{1+t^2}$.
2. $x=\dfrac{a(1-t^2)}{1+t^2}$, $y=\dfrac{2bt}{1+t^2}$.
3. $x=at^2$, $y=2at$.
4. $x=ct$, $y=\dfrac{c}{t}$.
5. $x=\dfrac{3at}{1+t^3}$, $y=\dfrac{3at^2}{1+t^3}$. *(K.U. Old Course, 1977)*

Find $\dfrac{dy}{dx}$ from :

6. $x^2+y^2+2gx+2fy+c=0.$
7. $x^{1/3}+y^{1/3}=a^{1/3}.$
8. $x^{2/3}+y^{2/3}=a^{2/3}.$
9. $x^3+y^3=3xy.$
10. $ax^2+2hxy+by^2=1.$
11. $x^n+y^n=a^n.$
12. $xy=c^2.$
13. $ax^2+2hxy+by^2+2gx+2fy+c=0.$ (*K.U. Nov. 1975*)

Differentiate :

14. x^n with respect to x^3.
15. $7x^5-12x^3$ w.r.t. $7x^2-15x$.
16. $\dfrac{x}{1+x^2}$ w.r.t. x^3.
17. $x-\sqrt{1-x^2}$ w.r.t. $\sqrt{1-x^2}$.
18. $\dfrac{ax+b}{cx+d}$ w.r.t. $\dfrac{a'x+b'}{c'x+d'}$.

Miscellaneous equations :

19. If $y=\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots\ldots\ldots}}}}$ to ∞

show that $(2y-1)\dfrac{dy}{dx}=1.$ (*K.U. 1977*)

20. If $y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\ldots\ldots\ldots}}}$ to ∞

prove that $\dfrac{dy}{dx}=\dfrac{y}{2y-x}.$

21. Assuming that

$$1+x+x^2+x^3+\ldots\ldots+x^n=\frac{1-x^{n+1}}{1-x},$$

find the sum of the series

$$1+2x+3x^2+\ldots\ldots+nx^{n-1} \text{ by differentiation.}$$

22. Assuming that

$$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+\ldots\ldots+{}^nC_nx^n$$

show that ${}^nC_1+2\,{}^nC_2+3\,{}^nC_3+\ldots\ldots+n\,{}^nC_n=n.2^{n-1}$.

[**Hint**. Differentiate the equation w.r.t. x and put $x=1$.]

23. If $y=\dfrac{x}{\sqrt{1-x}}$, prove that

$$x^3\frac{dy}{dx}=\left(1-\frac{x}{2}\right)y^3.$$

3

Differentiation of Circular and Inverse Circular Functions

3·1. In order to find the *Differential Coefficients* of circular and inverse circular functions, we have to make frequent use of the following important theorem on limits. This theorem has been proved in *Article 1·10.*

$$\underset{\theta \to 0}{\text{Lt}} \frac{\sin \theta}{\theta} = 1$$

where θ is measured in radians.

3·2. Differentiation of Circular Functions

(*i*) *Differential Coefficient of* $\sin x$.

Let $\quad y = \sin x$

Then, $\quad y + \delta y = \sin (x + \delta x)$

$\therefore \quad \delta y = \sin (x + \delta x) - \sin x$

$$= 2 \cos \left(x + \frac{\delta x}{2} \right) \sin \frac{\delta x}{2}$$

$$\left[\because \ \sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2} \right.$$

or $$\frac{\delta y}{\delta x} = \frac{2 \cos \left(x + \frac{\delta x}{2} \right) \sin \frac{\delta x}{2}}{\delta x}$$

$$= \cos \left(x + \frac{\delta x}{2} \right) \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}. \text{ (Please note this step)}$$

$\therefore$ $$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \left[\cos \left(x + \frac{\delta x}{2} \right) \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \right]$$

$$=\underset{\delta x\to 0}{\text{Lt}}\cos\left(x+\frac{\delta x}{2}\right)\times\underset{\delta x\to 0}{\text{Lt}}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

Now $\underset{\delta x\to 0}{\text{Lt}}\cos\left(x+\frac{\delta x}{2}\right)=\cos x$

and $\underset{\delta x\to 0}{\text{Lt}}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}=1$ (Proved above)

$\therefore$ $\frac{dy}{dx}=\cos x$

Hence $\frac{d}{dx}(\sin x)=\cos x.$

(*ii*) *Differential Coefficient of* $\cos x$.

Let $y=\cos x.$

Then $y+\delta y=\cos(x+\delta x)$

$\therefore$ $\delta y=\cos(x+\delta x)-\cos x$

$$=-2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

$$\left[\because \cos P-\cos Q=-2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}\right.$$

or $\frac{\delta y}{\delta x}=\frac{-2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x}$

$$=-\sin\left(x+\frac{\delta x}{2}\right)\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

(Please note this step)

$\because$ $\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}$

$$=\underset{\delta x\to 0}{\text{Lt}}\left[-\sin\left(x+\frac{\delta x}{2}\right)\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right]$$

$$=\underset{\delta x\to 0}{\mathrm{Lt}}\left[-\sin\left(x+\frac{\delta x}{2}\right)\right]\times\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}.$$

Now $\underset{\delta x\to 0}{\mathrm{Lt}}\left[-\sin\left(x+\frac{\delta x}{2}\right)\right]=-\sin x$

and $\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}=1$

$\therefore$ $\frac{dy}{dx}=-\sin x$

Hence $\frac{d}{dx}(\cos x)=-\sin x.$

(*iii*) *Differential coefficient of tan x.*

Let $y=\tan x$

Then $y+\delta y=\tan(x+\delta x).$

$$\therefore\quad \delta y=\tan(x+\delta x)-\tan x$$

$$=\frac{\sin(x+\delta x)}{\cos(x+\delta x)}-\frac{\sin x}{\cos x}\qquad\text{(Please note this step)}$$

$$=\frac{\sin(x+\delta x)\cos x-\cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}$$

$$=\frac{\sin(x+\delta x-x)}{\cos(x+\delta x)\cos x}$$

$[\because \sin A\cos B-\cos A\sin B=\sin(A-B)]$

$$=\frac{\sin\delta x}{\cos(x+\delta x)\cos x}$$

$$\therefore\quad \frac{\delta y}{\delta x}=\frac{1}{\cos(x+\delta x)\cos x}\cdot\frac{\sin\delta x}{\delta x}$$

$$\therefore\quad \frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta y}{\delta x}$$

$$=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{1}{\cos(x+\delta x)\cos x}\times\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\sin\delta x}{\delta x}$$

But $\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\sin x}{\delta x}=1$

and $$\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{1}{\cos(x+\delta x)\cos x}=\frac{1}{\cos^2 x}$$

$\therefore$ $$\frac{dy}{dx}=\frac{1}{\cos^2 x}=\sec^2 x.$$

Hence $$\frac{d}{dx}(\tan x)=\sec^2 x.$$

Alternative method (not by first principles)

Let $$y=\tan x=\frac{\sin x}{\cos x}$$

$\therefore$ $$\frac{dy}{dx}=\frac{\cos x\cdot\frac{d}{dx}(\sin x)-\sin x\cdot\frac{x}{dx}(\cos x)}{(\cos x)^2}$$

$$=\frac{\cos x\cdot\cos x-\sin x(-\sin x)}{\cos^2 x}$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x.$$

Hence $$\frac{d}{dx}(\tan x)=\sec^2 x.$$

(*iv*) *Differential coefficient of cot x.*

Let $$y=\cot x$$

then $$y+\delta y=\cot(x+\delta x)$$

$\therefore$ $$\delta y=\cot(x+\delta x)-\cot x$$

$$=\frac{\sin x\cos(x+\delta x)-\cos x\sin(x+\delta x)}{\sin(x+\delta x)\sin x)}$$

$$=\frac{\sin(x-\overline{x+\delta x})}{\sin(x+\delta x)\sin x}=\frac{\sin(-\delta x)}{\sin(x+\delta x)\sin x}$$

$$=\frac{-\sin\delta x}{\sin(x+\delta x)\sin x}\qquad[\because\ \sin(-\theta)=-\sin\theta$$

$\therefore$ $$\frac{\delta y}{\delta x}=-\frac{\sin\delta x}{\delta x}\cdot\frac{1}{\sin(x+\delta x)\sin x}$$

or $$\frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{\delta y}{\delta x}$$

$$=\underset{\delta x\to 0}{\mathrm{Lt}}\left(-\frac{\sin\delta x}{\delta x}\right)\times\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{1}{\sin(x+\delta x)\sin x}$$

$$=-1\times\frac{1}{\sin^2 x}=-\frac{1}{\sin^2 x}$$

$$=-\operatorname{cosec}^2 x$$

Hence $$\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x.$$

Alternative Method (not by *ab-initio*)

Let $$y=\cot x=\frac{\cos x}{\sin x}$$

$\therefore$ $$\frac{dy}{dx}=\frac{\sin x\times\frac{d}{dx}(\cos x)-\cos x\times\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$=\frac{\sin x(-\sin x)-\cos x(\cos x)}{\sin^3 x}$$

$$=-\frac{\sin^2 x+\cos^2 x}{\sin^2 x}=-\frac{1}{\sin^2 x}$$

$$=-\operatorname{cosec}^2 x.$$

(*v*) *Differential coefficient of* sec *x*.

Let $$y=\sec x$$

then $$y+\delta y=\sec(x+\delta x)$$

$\therefore$ $$\delta y=\sec(x+\delta x)-\sec x$$

$$=\frac{1}{\cos(x+\delta x)}-\frac{1}{\cos x}$$

(Please note this step)

$$=\frac{\cos x-\cos(x+\delta x)}{\cos(x+\delta x)\cos x}$$

$$=\frac{2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\cos(x+\delta x)\cos x}$$

$$\left[\because \cos P-\cos Q=2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}\right]$$

or $$\frac{\delta y}{\delta x}=\frac{2\sin\left(x+\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}\cdot\frac{\sin\frac{\delta x}{2}}{\delta x}$$

$$=\frac{\sin\left(x+\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}\cdot\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$\therefore$ $$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{dy}{dx}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\frac{\sin\left(x+\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}\times\underset{\delta x\to 0}{\text{Lt}}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$=\frac{\sin x}{\cos^2 x}\times 1=\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$$

$$=\sec x \tan x$$

Hence $\frac{d}{dx}(\sec x)=\sec x \tan x.$

Alternative Method (not by *ab-initio*)

Let $$y=\sec x=\frac{1}{\cos x}$$

$\therefore$ $$\frac{dy}{dx}=\frac{\cos x\times\frac{d}{dx}(1)-1\times\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$=\frac{\cos x\times 0-1\times(-\sin x)}{\cos^2 x}=\frac{\sin x}{\cos^2 x}$$

$$=\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}=\sec x \tan x.$$

(*vi*) *Differential coefficient of cosec x.* (*K.U. Nov., 1975*)

Let $$y=\text{cosec } x$$

then $$y+\delta y=\text{cosec }(x+\delta x)$$

$\therefore$ $$\delta y=\text{cosec }(x+\delta x)-\text{cosec } x$$

$$=\frac{1}{\sin(x+\delta x)}-\frac{1}{\sin x}$$

(Please note this step)

$$=\frac{\sin x-\sin(x+\delta x)}{\sin(x+\delta x)\sin x}$$

$$=\frac{2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{x-x+\delta x}{2}\right)}{\sin(x+\delta x)\sin x}$$

$$=\frac{2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(-\frac{\delta x}{2}\right)}{\sin(x+\delta x)\sin x}$$

$$=\frac{2\cos\left(x+\frac{\delta x}{2}\right)}{\sin(x+\delta x)\sin x}\times\left(-\sin\frac{\delta x}{2}\right)$$

[$\because$ $\sin(-\theta)=-\sin\theta$

or $$\frac{\delta y}{\delta x}=\frac{\cos\left(x+\frac{\delta x}{2}\right)}{\sin(x+\delta x)\sin \delta x}\times\left(-\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$

$$\therefore \qquad \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{\cos\left(x+\frac{\delta x}{2}\right)}{\sin^2(x+\delta x)\sin\delta x} \times \underset{\delta x \to 0}{\text{Lt}} \left(-\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$

$$= \frac{\cos x}{\sin x} \times -1 = -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$$

$$= -\text{cosec } x \cot x.$$

Hence

$$\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x.$$

Alternative Method (not by *ab-initio*)

Let $\qquad y = \text{cosec } x = \dfrac{1}{\sin x}$

$$\therefore \qquad \frac{dy}{dx} = \frac{\sin x \times \frac{d}{dx}(1) - 1 \times \frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x \times 0 - 1 \times \cos x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}$$

$$= -\text{cosec } x \cot x$$

Hence

$$\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x.$$

Note. (*i*) Alternative methods given above are not *ab-initio* methods. These have been given here for ambitious students.

(*ii*) *These results hold good only when x is measured in radians. These will not hold good if x is measured in degrees. In such a case, x is converted into radians before the result is applied.*

Example. If $y = \tan x^\circ$, find $\dfrac{dy}{dx}$.

Sol. Here we have, x degrees$= \dfrac{\pi x}{180}$ radians

$$\therefore \qquad y = \tan \frac{\pi x}{180}$$

$$\therefore \qquad \frac{dy}{dx}=\frac{\pi}{180}\sec^2\frac{\pi x}{180}$$

$$=\frac{\pi}{180}\sec^2 x^\circ.$$

Worked out Examples

Example 1. (*i*) Find $\frac{dy}{dx}$, if $y=\cos(3x+5)$.

Sol. Here $y=\cos(3x+5)$

Put $\qquad z=3x+5 \qquad \ldots(i)$

so that $\qquad y=\cos z \qquad \ldots(ii)$

Now $\qquad \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}$

From (*i*), we get

$$\frac{dy}{dx}=3, \text{ and}$$

from (*ii*), we get

$$\frac{dy}{dz}=-\sin z$$

$$=-\sin(3x+5) \qquad [\because \; z=3x+5$$

Hence $\qquad \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=-3\sin(3x+5).$

Alternative Method

$\cos(3x+5)$ is a function of $3x+5$

which is, in turn, a function of x. Hence $\cos(3x+5)$ is a function of a function.

$$\therefore \quad \frac{d}{dx}[\cos(3x+5)]=-\sin(3x+5)\times\frac{d}{dx}(3x+5)$$

$$=-3\sin(3x+5).$$

Example 2. Differentiate $\tan^2 x$ w.r.t. x.

Sol. Let $\quad y=\tan^3 x$

and $\qquad z=\tan x$, so that

$$y=z^3$$

Now $\qquad z=\tan x$

gives $\qquad \frac{dy}{dx}=\sec^2 x$

and $\qquad y=z^3$

gives $$\frac{dy}{dz}=3z^2=3\tan^2 x.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=3\tan^2 x\ .\ \sec^2 x.$$

Alternative Method

Let $$y=\tan^2 x=(\tan x)^3$$

$$\therefore \quad \frac{dy}{dx}=3\,(\tan x)^2\times\frac{d}{dx}\,(\tan x)$$

$$=3\tan^2 x\ .\ \sec^2 x.$$

Note. The student is advised to *completely* master these *Alternative* methods, as he can save a lot of time by adopting these methods while solving questions on differentiation. He is advised to note the following examples in particular :

Ex. (*i*) $\frac{d}{dx}\,(\sin mx)=m\cos mx.$

(*ii*) $\frac{d}{dx}\,(\sin^m x)=m\sin^{m-1} x\ .\ \cos x.$

(*iii*) $\frac{d}{dx}\,(\sin^m nx)=mn\sin^{m-1} nx\ .\ \cos nx.$

Example 3. Differentiate $\tan mx$ *ab-initio.*

Sol. Let $$y=\tan mx$$

then $$y+\delta y=\tan m\,(x+\delta x)$$

$$\therefore \quad \delta y=\tan m\,(x+\delta x)-\tan mx$$

$$=\frac{\sin m\,(x+\delta x)}{\cos m\,(x+\delta x)}-\frac{\sin mx}{\cos mx}$$

$$=\frac{\sin m\,(x+\delta x)\cos mx-\cos m(x+\delta x)\sin mx}{\cos m\,(x+\delta x)\cos mx}$$

$$=\frac{\sin\,[m\,(x+\delta x)-mx]}{\cos m(x+\delta x)+\cos mx}$$

$$=\frac{\sin m\,\delta x}{\cos m(x+\delta x)\cos mx}$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{\dfrac{\sin m\,\delta x}{\delta x}}{\cos m\,(x+\delta x)\cos mx}$$

$$=\frac{m\times\dfrac{\sin m\,\delta x}{m\,\delta x}}{\cos m\,(x+\delta x)\cos mx}$$

$$\therefore \qquad \frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\delta y}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\dfrac{m\sin m\,\delta x}{m\,\delta x}}{\cos m(x+\delta x)\cos mx}$$

$$=\frac{m}{\cos^2 mx}=m\sec^2 mx.$$

Example 4. Differentiate $\sin x^2$ *from first principles.*

Sol. Let $\qquad y=\sin x^2$

then $\qquad y+\delta y=\sin(x+\delta x)^2$

$$\therefore \qquad \delta y=\sin(x+\delta x)^2-\sin x^2$$

$$=2\cos\left[\frac{(x+\delta x)^2+x^2}{2}\right]\times\sin\left[\frac{(x+\delta x)^2-x^2}{2}\right]$$

or

$$\frac{\delta y}{\delta x}=\cos\left[\frac{(x+\delta x)^2+x^2}{2}\right]\times\frac{\sin\left[\dfrac{(x+\delta x)^2-x^2}{2}\right]}{\dfrac{(x+\delta x)^2-x^2}{2}}\times\frac{(x+\delta x)^2-x^2}{\delta x}$$

Now $\qquad \underset{\delta x\to 0}{\text{Lt}}\ \cos\left[\dfrac{(x+\delta x)^2+x^2}{2}\right]=\cos x^2$

and $\qquad \underset{\delta x\to 0}{\text{Lt}}\ \dfrac{\sin\left[\dfrac{(x+\delta x)^2-x^2}{2}\right]}{\dfrac{(x+\delta x)^2-x^2}{2}}=1$

Also $\qquad \underset{\delta x\to 0}{\text{Lt}}\ \dfrac{(x+\delta x)^2-x^2}{\delta x}$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{x^2\left(1+\dfrac{\delta x}{x}\right)^2-x^2}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{x^2\left[1+2\dfrac{\delta x}{x}+\dfrac{2(2-1)}{2\,!}\left(\dfrac{\delta x}{x}\right)^2+\ldots\right]-x^2}{\delta x}$$

$$=2x$$

Hence $\qquad \dfrac{dy}{dx}=2x\cos x^2.$

Example 5. Differentiate $\sin^{-1} 2x\sqrt{1-x^2}$ w.r.t. x.

Sol. Let $\qquad y=\sin^{-1} 2x\sqrt{1-x^2}$ $\qquad$ ...(1)

Put $\qquad x=\sin\theta$

so that
$$y=\sin^{-1}(2\sin\theta\ \sqrt{1-\sin^2\theta})$$
$$=\sin^{-1}(2\sin\theta\cos\theta)$$
$$=\sin^{-1}\sin 2\theta$$
$$=2\theta. \qquad ...(2)$$

Now (1) gives

$\dfrac{dx}{d\theta}=\cos\theta$, and (2) gives

$$\frac{dy}{d\theta}=2.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$$
$$=2\times\frac{1}{\cos\theta}=\frac{2}{\sqrt{1-\sin^2\theta}}$$
$$=\frac{2}{\sqrt{1-x^2}}.$$

Example 6. Given $\sin 3x=3\sin x-4\sin^3 x$; show, by differentiation, that

$$\cos 3x=4\cos^3 x-3\cos x.$$

Sol. $\sin 3x=3\cos x-4\sin^3 x.$

Differentiating both sides w.r.t. x we get

$$3\cos 3x=3\cos x-3\times 4\sin^2 x\cos x$$

or
$$\cos 3x=\cos x-4(1-\cos^2 x)\cos x$$

$$\therefore \quad \cos 3x=4\cos^3 x-3\cos x.$$

Example 7. If $y=A\sin mx+B\cos mx$, prove that

$$\frac{d^2y}{dx^2}+m^2y=0. \qquad \text{(P.U.)}$$

Sol. $y=A\sin mx+B\cos mx$

is
$$\frac{dy}{dx}=Am\cos mx-Bm\sin mx$$

and
$$\frac{d^2y}{dx^2}=-Am^2\sin mx-Bm^2\cos mx$$
$$=-m(A\sin mx+B\cos mx).$$

But $A\sin mx+B\cos mx=y$

$$\therefore \quad \frac{d^2y}{dx^2}=-m^2y$$

which gives
$$\frac{d^2y}{dx^2}+m^2y=0.$$

Example 8. Differentiate $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ w.r.t.

$$\sec^{-1}\frac{1}{2x^2-1}.$$

Sol. Put $\quad x=\cos\theta$, so that

$$\tan^{-1}\frac{x}{\sqrt{1-x^2}}=\tan^{-1}\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}=\tan^{-1}\frac{\cos\theta}{\sin\theta}$$

$$=\tan^{-1}\cot\theta=\tan^{-1}\tan\left(\frac{\pi}{2}-\theta\right)$$

$$=\frac{\pi}{2}-\theta$$

and $\quad \sec^{-1}\dfrac{1}{2x^2-1}=\sec^{-1}\dfrac{1}{2\cos^2\theta-1}=\sec^{-1}\dfrac{1}{\cos 2\theta}$

$$=\sec^{-1}\sec 2\theta=2\theta$$

Now if we put $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}=u$, then

$$u=\frac{\pi}{2}=\theta$$

Similarly, putting $z=\sec^{-1}\dfrac{1}{2x^2-1}$, we have

$$z=2\theta$$

We have to differentiate $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ w.r.t. $\sec^{-1}\dfrac{1}{2x^2-1}$ which means that we want $\dfrac{du}{dz}$ which is, in turn, equal to $\dfrac{du}{d\theta}\times\dfrac{d\theta}{dz}=-1\times\dfrac{1}{2}=-\dfrac{1}{2}$.

$\left[\because\right.$ from $u=\dfrac{\pi}{2}=\theta$, we get $\dfrac{du}{d\theta}=-1$, and from $z=2\theta$, $\left.\dfrac{dz}{d\theta}=2\right]$

Exercise VII

I. Differentiate the following functions w.r.t. x :

1. $\cos mx$.
2. $\sin x^{\circ}$.
3. $\cos^m x$.
4. $\tan^3(3x+6)$.
5. $\sin^4(ax+b)^2$. (*K.U. 1975*)
6. $\tan(\sin x)$.
7. $x\sec^2 x-\tan x$.
8. $\sin^m x\cos^n x$.
9. $\dfrac{1-\sin x}{1+\sin x}$.

10. $\dfrac{a+b\cos x}{b+a\cos x}$.

II. Find $\dfrac{dy}{dx}$, when

1. $y=\sqrt{\dfrac{1+\cos x}{1-\cos x}}$.

2. $y=\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$. (*K.U. 1976*)

3. $y=\tan^{-1}\left(\dfrac{\sin x+\cos x}{\cos x-\sin x}\right)$. (*K.U. Nov. 1975, 1977*)

4. $y=\dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$.

5. $y=\tan^{-1}(\sec x+\tan x)$.

6. $y=\cot^{-1}\sqrt{\dfrac{1+\cos 3x}{1-\cos 3x}}$ (*P.U. 1957*)

7. $y=\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$.

8. $y=\tan^{-1}\sqrt{\dfrac{1+\cos x}{1-\cos x}}$.

$\left[\textbf{Hint.}\ \cos x=\dfrac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}}\ \text{etc.}\right]$

III. Differentiate the following w.r.t. x :

1. $\sin^{-1}\dfrac{2x}{1+x^2}$.

2. $\cos^{-1}\dfrac{1-x^2}{1+x^2}$

3. $\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$.

4. $\sin^{-1}(3x-4x^3)$.

5. $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-1}{x}\right]$.

6. $\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$.

7. $\tan^{-1}\dfrac{1-x^2}{1+x^2}$. (*K.U. 1975*)

IV. Differentiate :

1. $\tan x$ w.r.t. $\sin x$.

2. $\sin x^4$ w.r.t. x^2.

3. $\sin x$ w.r.t. $\cos x$.

4. $\sin x^3$ w.r.t. $\sin^3 x$.

5. $\tan^{-1}\dfrac{2x}{1-x^2}$ w.r.t. $\sin^{-1}\dfrac{2x}{1+x^2}$.

6. $\sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$ w.r.t. $\cos^{-1}\dfrac{1-x^2}{1+x^2}$.

7. $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-1}{x}\right]$ w.r.t. $\tan^{-1}x$.

8. $\sec^{-1} \dfrac{1}{2x^2-1}$ w.r.t. $\sqrt{1+x^2}$. (K.U. 1975)

V. Differentiate :

1. $\sqrt{\sin x}$.   2. $\sqrt{\sin \sqrt{x}}$.

3. $\sqrt{\tan \sqrt{x}}$.   4. $\sqrt{\tan \sqrt{1+x^2}}$.

VI. If $\sin y = x \sin (a+y)$, prove that

$$\frac{dy}{dx} = \frac{\sin^2 (a+y)}{\sin a}.$$ (K.U. 1976)

VII. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x \ldots \text{to } \infty}}}$

show that $$\frac{dy}{dx} = \frac{\cos x}{2y-1}.$$

VIII. Find $\dfrac{dy}{dx}$, when :

1. $x = a \cos \theta,\ y = b \sin \theta$.
2. $x = a \cos^3 \theta,\ y = b \sin^3 \theta$.
3. $x = a \sec^3 \theta,\ y = b \tan^3 \theta$.
4. $x = a (\theta + \sin \theta),\ y = a (1 - \cos \theta)$.

IX. Differentiate *ab-initio* :

1. $\cos 3x$.   2. $\sec 5x$.
3. $\sin \sqrt{x}$. (K.U. 1976)
4. $\cos^2 x$.   5. $\sqrt{\cos x}$. (K.U. 1975)
6. $\tan x^2$.   7. $x \sin x$. (K.U. 1975)
8. $\sqrt{\tan x}$. (K.U. 1975)

3·3. Inverse Circular Functions

Def. If $\sin x = y$, then x is such an angle whose sine is equal to y. This can be expressed by saying that $\sin^{-1} y = x$.

Again, if $\sin x = \dfrac{\sqrt{3}}{2}$ then $x = \sin^{-1} \dfrac{\sqrt{3}}{2}$ and we know that there are many angles whose sin is $\dfrac{\sqrt{3}}{2}$. These are $\dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{7\pi}{3}$, etc. This shows that $\sin^{-1} x$ (and similarly any other Inverse Circular function) is many-valued. But we are concerned here only with the smallest value, so far as differentiation is concerned. This smallest value is known as the *Principal Value* of the inverse function.

3·4. Differentiation of Inverse Circular Functions

(*i*) *Differential Coefficient of* $\sin^{-1} x$.

Let $y = \sin^{-1} x \quad \therefore \quad x = \sin y$ ...(1)

then $y + \delta y = \sin^{-1} (x + \delta x)$

or $x + \delta x = \sin (y + \delta y)$ ...(2)

Subtracting (1) from (2), we have

$$\delta x=\sin(y+\delta y)-\sin y$$
$$=2\cos\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2}$$

or
$$\frac{\delta x}{\delta y}=\cos\left(y+\frac{\delta y}{2}\right)\frac{\sin\frac{\delta y}{2}}{\frac{\delta y}{2}}$$

$$\therefore\quad \frac{dx}{dy}=\underset{\delta y\to 0}{\mathrm{Lt}}\frac{\delta x}{\delta y}=\underset{\delta y\to 0}{\mathrm{Lt}}\left[\cos\left(y+\frac{\delta y}{2}\right)\frac{\sin\frac{\delta y}{2}}{\frac{\delta y}{2}}\right]$$
$$=\cos y$$

or
$$\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$$

Hence $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$.

Alternative Method

Let $\quad y=\sin^{-1}x$

$\therefore \quad x=\sin y$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy}=\cos y$$

or
$$\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}.$$

Hence $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$.

(*ii*) *Differential Coefficient of $\cos^{-1}x$.*

Let $\quad y=\cos^{-1}x$

$\therefore \quad x=\cos y \quad \ldots(1)$

Now $\quad y+\delta y=\cos^{-1}(x+\delta x)$

or $\quad x+\delta x=\cos(y+\delta y) \quad \ldots(2)$

Subtracting (1) from (2), we get

$$\delta x=\cos(y+\delta y)-\cos y$$
$$=-2\sin\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2}$$

or
$$\frac{\delta x}{\delta y}=-\sin\left(y+\frac{\delta y}{2}\right)\frac{\sin\frac{\delta y}{2}}{\frac{\delta y}{2}}$$

$$\therefore \quad \frac{dx}{dy}=\underset{\delta y\to 0}{\text{Lt}}\frac{\delta x}{\delta y}=\underset{\delta y\to 0}{\text{Lt}}\left[-\sin\left(y+\frac{\delta y}{2}\right)\frac{\sin\frac{\delta y}{2}}{\frac{\delta y}{2}}\right]$$

$$=-\sin y$$

or $$\frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-\cos^2 y}}=-\frac{1}{\sqrt{1-x^2}}$$

Hence $$\frac{d}{dx}(\cos^{-1} x)=-\frac{1}{\sqrt{1-x^2}}.$$

Alternative Method

Let $y=\cos^{-1} x$

$\therefore$ $x=\cos y.$

Differentiating this w.r.t. y, we get

$$\frac{dx}{dy}=\sin y$$

or $$\frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-\cos^2 y}}$$

$$=-\frac{1}{\sqrt{1-x^2}}$$

Hence $$\frac{d}{dx}(\cos^{-1} x)=-\frac{1}{\sqrt{1-x^2}}.$$

(*iii*) *Differential Coefficient of* $\tan^{-1} x$.

Let $y=\tan^{-1} x$

$\therefore$ $x=\tan y$ ...(1)

Now $y+\delta y=\tan^{-1}(x+\delta x)$

or $x+\delta x=\tan(y+\delta y)$ ...(2)

Subtracting (1) from (2), we have

$$\delta x=\tan(y+\delta y)-\tan y$$

$$=\frac{\sin(y+\delta y)}{\cos(y+\delta y)}-\frac{\sin y}{\cos y}$$

$$=\frac{\sin(y+\delta y)\cos y-\cos(y+\delta y)\sin y}{\cos(y+\delta y)\cos y}$$

$$=\frac{\sin(y+\delta y-y)}{\cos(y+\delta y)\cos y}=\frac{\sin\delta y}{\cos(y+\delta y)\cos y}$$

or $$\frac{\delta x}{\delta y}=\frac{\frac{\sin\delta y}{\delta y}}{\cos(y+\delta y)\cos y}$$ [Please note this]

$$\therefore \quad \frac{dx}{dy}=\underset{\delta y\to 0}{\text{Lt}}\frac{\frac{\sin\delta y}{\delta y}}{\cos(y+\delta y)\cos y}=\frac{1}{\cos^2 y}$$

$$=\sec^2 y$$

or $$\frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}$$

Hence $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$.

Alternative Method

Let $$y=\tan^{-1} x$$

$\therefore$ $$x=\tan y$$

Differentiating this w.r.t. y, we have

$$\frac{dx}{dy}=\sec^2 y$$

$\therefore$ $$\frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}$$

Hence $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$.

(*iv*) *Differential Coefficient of* $\cot^{-1} x$.

Let $y=\cot^{-1} x$

$\therefore$ $x=\cot y$ ...(1)

Now $y+\delta y=\cot^{-1}(x+\delta x)$

or $x+\delta x=\cot(y+\delta y)$ ...(2)

Subtracting (1) from (2), we get

$$\begin{aligned}\delta x&=\cot(y+\delta y)-\cot y\\&=\frac{\cos(y+\delta y)}{\sin(y+\delta y)}-\frac{\cos y}{\sin y}\\&=\frac{\sin y\cos(y+\delta y)-\cos y\sin(y+\delta y)}{\sin(y+\delta y)\sin y}\\&=\frac{\sin(y-\overline{y+\delta y})}{\sin(y+\delta y)\sin y}=\frac{-\sin\delta y}{\sin(y+\delta y)\sin y}\end{aligned}$$

or $$\frac{\delta x}{\delta y}=\frac{\frac{-\sin\delta y}{\delta y}}{\sin(y+\delta y)\sin y}$$

$\therefore$ $$\frac{dx}{dy}=\underset{\delta y\to 0}{\mathrm{Lt}}\frac{\delta x}{\delta y}=\underset{\delta y\to 0}{\mathrm{Lt}}\frac{\frac{-\sin\delta y}{\delta y}}{\sin(y+\delta y)\sin y}$$

$$=-\frac{1}{\sin^2 y}=-\operatorname{cosec}^2 y$$

or $$\frac{dy}{dx}=-\frac{1}{\operatorname{cosec}^2 y}=-\frac{1}{1+\cot^2 y}=-\frac{1}{1-x^2}$$

Hence $$\frac{d}{dx}(\cot^{-1} x)=-\frac{1}{1+x^2}.$$

Alternative Method

Let $y = \cot^{-1} x$

$\therefore$ $x = \cot y$

Differentiating this w.r.t. y, we get

$$\frac{dx}{dy} = -\operatorname{cosec}^2 y$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\operatorname{cosec}^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}.$$

Hence $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}.$

(v) *Differential Coefficient of* $\sec^{-1} x$.

Let $y = \sec^{-1} x$

$\therefore$ $x = \sec y$ ...(1)

Now $y + \delta y = \sec^{-1}(x + \delta x)$

or $x + \delta x = \sec(y + \delta y)$ ...(2)

Subtracting (1) from (2), we get

$$\delta x = \sec(y+\delta y) - \sec y$$

$$= \frac{1}{\cos(y+\delta y)} - \frac{1}{\cos y} = \frac{\cos y - \cos(y+\delta y)}{\cos(y+\delta y)\cos y}$$

$$= \frac{2\sin\left(y + \frac{\delta y}{2}\right)\sin\frac{\delta y}{2}}{\cos(y+\delta y)\cos y}$$

or

$$\frac{\delta x}{\delta y} = \frac{\sin\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2} \Big/ \frac{\delta y}{2}}{\cos(y+\delta y)\cos y}$$

or

$$\frac{dx}{dy} = \underset{\delta y \to 0}{\mathrm{Lt}} \frac{\sin\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2} \Big/ \frac{\delta y}{2}}{\cos(y+\delta y)\cos y}$$

$$= \frac{\sin y}{\cos y} \cdot \frac{1}{\cos y} = \sec y \tan y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$= \frac{1}{\sec y \sqrt{\sec^2 y - 1}}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$

Hence $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}.$

Alternative Method

Let $y = \sec^{-1} x$

$\therefore$ $x = \sec y$

Differentiating this w.r.t. y, we get

$$\frac{dx}{dy}=\sec y \tan y.$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\sec y \tan y}=\frac{1}{\sec y\sqrt{\sec^2 y-1}}$$

$$=\frac{1}{x\sqrt{x^2-1}}$$

Hence $$\frac{d}{dx}(\sec^{-1} x)=\frac{1}{x\sqrt{x^2-1}}.$$

(*vi*) *Differential coffiecient of* $cosec^{-1}\, x$.

Let $y=\operatorname{cosec}^{-1} x$

$\therefore \quad x=\operatorname{cosec} y \qquad \ldots(1)$

Now $y+\delta y=\operatorname{cosec}^{-1}(x+\delta x)$

or $x+\delta x=\operatorname{cosec}(y+\delta y) \qquad \ldots(2)$

Subtracting (1) from (2), we get

$$\delta x=\operatorname{cosec}(y+\delta y)-\operatorname{cosec} y$$

$$=\frac{1}{\sin(y+\delta y)}-\frac{1}{\sin y}$$

$$=\frac{\sin y-\sin(y+\delta y)}{\sin(y+\delta y)\sin y}$$

$$=\frac{2\cos\left(y+\frac{\delta y}{2}\right)\sin\left(-\frac{\delta y}{2}\right)}{\sin(y+\delta y)\sin y}$$

$$=\frac{-2\cos\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2}}{\sin(y+\delta y)\sin y}$$

or $$\frac{dx}{dy}=\frac{-\cos\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2}\Big/\frac{\delta y}{2}}{\sin(y+\delta y)\sin y}$$

$$\therefore \quad \frac{dx}{dy}=\underset{\delta y\to 0}{\text{Lt}}\frac{\delta x}{\delta y}$$

$$=\underset{\delta y\to 0}{\text{Lt}}\frac{-\cos\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2}\Big/\frac{\delta y}{2}}{\sin(y+\delta y)\sin y}$$

$$=-\frac{\cos y}{\sin y}\cdot\frac{1}{\sin y}$$

$$=-\cot y \operatorname{cosec} y$$

or
$$\frac{dy}{dx}=-\frac{1}{\operatorname{cosec} y \cot y}$$
$$=-\frac{1}{\operatorname{cosec} y \sqrt{\operatorname{cosec}^2 y-1}}$$
$$=-\frac{1}{x\sqrt{x^2-1}}.$$

Hence $$\frac{d}{dx}(\operatorname{cosec}^{-1} x)=-\frac{1}{x\sqrt{x^2-1}}$$

Alternative Method

Let $y=\operatorname{cosec}^{-1} x$

$\therefore$ $x=\operatorname{cosec} y$

Differentiating this w.r.t. y, we get
$$\frac{dx}{dy}=-\operatorname{cosec} y \cot y.$$
$\therefore$
$$\frac{dy}{dx}=-\frac{1}{\operatorname{cosec} y \cot y}$$
$$=-\frac{1}{\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 y-1}}$$
$$=-\frac{1}{x\sqrt{x^2-1}}$$

Hence $$\frac{d}{dx}(\operatorname{cosec}^{-1} x)=-\frac{1}{x\sqrt{x^2-1}}$$

Solved Examples

Example 1. If
$$y=\cos^{-1}(\sin x), \text{ find } \frac{dy}{dx}.$$

Sol. Let $z=\sin x$ ...(*i*)

so that $y=\cos^{-1} z$ ...(*ii*)

Now from (*i*) $\frac{dz}{dx}=\cos x$

and from (*ii*)
$$\frac{dy}{dz}=-\frac{1}{\sqrt{1-z^2}}=-\frac{1}{\sqrt{1-\sin^2 x}}=-\frac{1}{\cos x}$$
$\therefore$
$$\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}$$
$$=-\frac{1}{\cos x}\times\cos x=-1.$$

Example 2. Differentiate $\sin^{-1}\frac{a+b\cos x}{b+a\cos x}$.

Sol. Let $y=\sin^{-1}\frac{a+b\cos x}{b+a\cos x}$

and $$z=\frac{a+b\cos x}{b+a\cos x} \qquad \ldots(i)$$

so that $$y=\sin^{-1} z. \qquad \ldots(ii)$$

Now form (i)

$$\frac{dz}{dx}=\frac{-b\sin x\,(b+a\cos x)+a\sin x(a+b\cos x)}{(b+a\cos x)^2}$$

$$=\frac{-(b^2-a^2)\sin x}{(b+a\cos x)^2}$$

and from (ii) $$\frac{dy}{dz}=\frac{1}{\sqrt{1-z^2}}$$

$$=\frac{1}{\sqrt{1-\left(\dfrac{a+b\cos x}{b+a\cos x}\right)^2}}$$

$$=\frac{b+a\cos x}{\sqrt{(b+a\cos x)^2(-a+b\cos x)^2}}$$

$$=\frac{b+a\cos x}{\sqrt{(b^2-a^2)-(b^2-a^2)\cos^2 x}}$$

$$=\frac{b+a\cos x}{\sqrt{b^2-a^2}\sin x}$$

$\therefore$ $$\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}$$

$$=\frac{b+a\cos x}{\sqrt{b^2-a^2}\sin x}\times-\frac{(b^2-a^2)\sin x}{(b+a\cos x)^2}$$

$$=-\frac{\sqrt{b^2-a^2}}{b+a\cos x}.$$

Example 3. If

$$y=\frac{\sin^{-1} x}{\sqrt{1-x^2}}, \text{ show that}$$

$$(1-x^2)\frac{dy}{dx}=1+xy.$$

Sol. $$y=\frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

$\therefore$ $y^2(1+x^2)=(\sin^{-1} x)^2.$ (Please note this step)

Differentiating both sides w.r.t. x, we get

$$2xy_1(1-x^2)-2xy^2=2\frac{\sin^{-1} x}{\sqrt{1-x^2}}=2y \qquad \text{(Please note this step)}$$

or $y_1(1-x^2)-xy=1.$

Hence $(1-x^2)\dfrac{dy}{dx}=1+xy.$

Exercise VIII

I. Differentiate the following w.r.t. x :

1. $\sin^{-1} 3x$. 2. $\cos^{-1}(3x+2)$. 3. $\frac{1}{a}\tan^{-1}\frac{x}{a}$.

4. $\sec^{-1} x^2$. 5. $\tan^{-1}(\tan x)$. 6. $\tan(\tan^{-1} x)$.

7. $\operatorname{cosec}^{-1}\sqrt{x}$. 8. $\tan^{-1} x+\tan^{-1}\frac{1}{x}$.

9. $\tan x.\tan^{-1} x$. 10. $\cos^{-1}\frac{b+a\cos x}{a+b\cos x}$. *(K.U. March 1975)*

II. Differentiate :

1. $\sin^{-1}\frac{1}{x}$ w.r.t. $\tan^{-1}(2x+3)$.

2. $\sin^{-1}\frac{x}{\sqrt{1+x^2}}$ w.r.t. $\cos^{-1}\frac{1-x^2}{1+x^2}$.

3. $\tan^{-1}\frac{2x}{1-x^2}$ w.r.t. $\sin^{-1}\frac{2x}{1+x^2}$. *(K.U. 1976)*

III. 1. Differentiate both sides to show that :

(*i*) $3\sin^{-1} x=\sin^{-1}(3x-4x^3)$.

(*ii*) $2\tan^{-1} x=\tan^{-1}\frac{2x}{1-x^2}$.

2. If $y=\sin^{-1} x+\sin^{-1}\sqrt{1-x^2}$, prove that

$$\frac{dy}{dx}=0. \quad \left[\textbf{Hint.} \quad \sin^{-1}\sqrt{1-x^2}=\cos^{-1} x\right]$$

3. If $y=\sin^{-1} x$, prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}.$$

4. If $y=\cos(m\sin^{-1} x)$, so that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0.$$

IV. Differentiate w.r.t. x :

1. $\sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$. [**Hint.** Put $x=\cos\theta$.]

2. $\sin^{-1}\frac{\sqrt{1+x}+\sqrt{1-x}}{2}$. [**Hint.** Put $x=\cos\theta$.]

3. $\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right]$.

4. $\frac{x\cos^{-1} x}{\sqrt{1-x^2}}$. *(P.U. 1942)*

V. Show by differentiation that

$$\tan^{-1}\frac{1-x^2}{2x}+\sin^{-1}\frac{2x}{1+x^2} \text{ is constant.}$$ *(K.U. Nov. 1975)*

[**Hint.** Put $x=\tan\theta$].

4

Differentiation of Logarithmic, Exponential and Hyperbolic Functions

4·1. Before attempting to differentiate logarithmic, exponential and hyperbolic functions, we shall find the limit of $\left(1+\frac{1}{n}\right)^n$ as n tends to infinity. With the help of this limit, we shall next proceed to find the limits of $(1+x)^{1/x}$ and $\frac{a^x-1}{x}$ as x tends to zero. These two limits will enable us to differentiate the functions referred to above.

4·2. To show that $\underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n=e$ *(K.U. 1977)*

where $$2<e<3.$$

Expanding this by the Binomial Theorem, we have

$$\left(1+\frac{1}{n}\right)^n=1+n.\frac{1}{n}+\frac{n(n-1)}{2\,!}\cdot\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3\,!}\left(\frac{1}{n}\right)^3+\ldots\ldots$$

$$=1+1+\frac{1}{2\,!}\left(1-\frac{1}{n}\right)+\frac{1}{3\,!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots\ldots$$

As $n\to\infty$, $\frac{1}{n}$, $\frac{2}{n}$,......, all $\to 0$.

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n=1+1+\frac{1}{2\,!}+\frac{1}{3\,!}+\ldots\ldots\text{to }\infty$$

$=2{\cdot}7182818=2{\cdot}72$ to two places of decimals.

Now we put $$e=2{\cdot}72$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n=e \text{ where } 2<e<3.$$

4·21. To show that $\underset{x\to 0}{\text{Lt}}\ (1+x)^{1/x}=e$ where $2<e<3$.

Put $\qquad n=\dfrac{1}{x}$, so that $x\to 0$ as $n\to\infty$

$\therefore \qquad \underset{n\to 0}{\text{Lt}}\ (1+x)^{1/x}=\underset{x\to 0}{\text{Lt}}\left(1+\dfrac{1}{n}\right)^n=e$ (Already proved)

4·22. To show that $\underset{x\to 0}{\text{Lt}}\ \dfrac{a^x-1}{x}=\log a.$ *(K.U. Nov. 1976)*

Let $\quad a^x-1=z$, so that $z\to 0$ as $x\to 0$

Now $\qquad a^x-1=z$

$\therefore \qquad a^x=1+z$

or $\qquad \log a^x=\log(1+z)$

or $\qquad x\log a=\log(1+z)$

$$x=\frac{\log(1+z)}{\log a}.$$

$$\therefore \quad \underset{n\to 0}{\text{Lt}}\ \frac{a^x-1}{x}=\underset{z\to 0}{\text{Lt}}\ \frac{z}{\dfrac{\log(1+z)}{\log a}}=\underset{z\to 0}{\text{Lt}}\ \frac{z\log a}{\log(1+z)}$$

$$=\frac{\log a}{\underset{z\to 0}{\text{Lt}}\ \frac{1}{z}\log(1+z)}=\frac{\log a}{\underset{z\to 0}{\text{Lt}}\ \log(1+z)^{1/z}}$$

$$=\frac{\log a}{\log \underset{z\to 0}{\text{Lt}}\ (1+z)^{1/z}}=\frac{\log a}{\log e}=\log a.$$

Cor. $\underset{n\to 0}{\text{Lt}}\ \dfrac{e^x-1}{x}=1.$ *(K.U. 1977)*

Put $\qquad a=e$, then $\log e=1$.

Or

$$\underset{n\to 0}{\text{Lt}}\left(\frac{e^x-1}{x}\right) \qquad \ldots(i)$$

Put $\qquad e^x-1=z$

$\therefore \qquad e^x=1+z$

$\therefore \qquad z\to 0$ as $x\to 0$

$\therefore \qquad \log e^x=\log(1+z)$

or $\qquad x=\log(1+z)$

or (*i*) becomes :

$$\underset{z\to 0}{\text{Lt}} \frac{z}{\log(1+z)} = \underset{z\to 0}{\text{Lt}} \frac{1}{\frac{1}{z}\log(1+z)}$$

$$= \underset{z\to 0}{\text{Lt}} \frac{1}{\log(1+z)^{\frac{1}{z}}} = \frac{1}{\log e} = 1.$$

Note. In the foregoing article, we have applied an important formula of Trigonometry pertaining to logarithm. The formula is : $\log_a m^n = n \log_a m$. The student is advised to note this carefully. He is also advised to note that :

(1) $\log_a mn = \log_a m + \log_a n$.

(2) $\log_a m/n = \log_a m - \log_a n$.

(3) $\log_a m = \log_a m \times \log_a b = \dfrac{\log b^m}{\log b^a}$.

(4) $e^{\log x} = x$. (5) $\log_a a = 1$.

4·3. Napier Logarithms

In Trigonometry, we take 10 as the base of logarithms. In Calculus, however, we take e as the base, so that when no base is mentioned we take it to be e.

Thus $\log x$ always means $\log_e x$.

Logarithms with e as their base are known as *Napiers Logarithms.*

4·4. Find the Differential coefficient of $\log_a x$.

Let $y = \log_a x$

then $y + \delta y = \log_a (x + \delta x)$

or $\delta y = \log_a (x+\delta x) - \log_a x$

$$= \log_a \frac{x+\delta x}{x}$$

$$\left(\because \log_a m - \log_a n = \log_a \frac{m}{n} \right)$$

$$= \log_a \left(1 + \frac{\delta x}{x} \right)$$

or $$\frac{\delta y}{\delta x} = \frac{1}{\delta x} \log_a \left(1 + \frac{\delta x}{x} \right)$$

$$=\frac{1}{x}\cdot\frac{x}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$=\frac{1}{x}\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

[Please note these two steps]

$$\therefore \quad \frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}=\frac{1}{x}\underset{\delta x\to 0}{\text{Lt}}\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

$$=\frac{1}{x}\log_a\underset{\delta x\to 0}{\text{Lt}}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

Now $\underset{\delta x\to 0}{\text{Lt}}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}=e \quad \left[\because \underset{x\to 0}{\text{Lt}}(1+x)^{\frac{1}{x}}=e\right]$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{x}\log_a e$$

Hence $\frac{d}{dx}(\log_a x)=\frac{1}{x}\log_a e.$

Cor. If $a=e$, then $\frac{d}{dx}(\log x)=\frac{1}{x}\log_e e$

$$=\frac{1}{x}.$$

Note. *The Cor. under article 4·4, gives us a very important working rule differentiating Logarithmic Functions. The rule can be laid down as under :*

Differential coefficient of the log (of some function of x)

$=\frac{1}{\textit{that function of } x}\times$ *Differential coefficient (of that function of x) w.r.t. x.*

This can be illustrated by means of the following solved examples.

Example 1. Differentiate log $3x$ with respect to x.

Sol. Let $y=\log 3x$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{3x}\times\text{Diff. coefficient of }(3x)$$

$$=\frac{3}{3x}=\frac{1}{x}.$$

Alternative Method :

Let $y=\log 3x$ and $z=3x$

so that $y=\log z$.

Now $y=\log z$ gives $\dfrac{dy}{dz}=\dfrac{1}{z}=\dfrac{1}{3x}$

and $z=3x$ gives $\dfrac{dz}{dx}=3$.

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=\frac{1}{3x}\times 3=\frac{1}{x}.$$

Example 2. Differentiate $\log(\sin x)$ w.r.t. x.

Sol. Let $y=\log(\sin x)$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\sin x}\times\frac{d}{dx}(\sin x)$$

$$=\frac{\cos x}{\sin x}=\cot x.$$

Example 3. If $y=\log\tan\dfrac{x}{2}$, show that

$$\frac{dy}{dx}=\operatorname{cosec} x.$$

Sol. Here $y=\log\tan\dfrac{x}{2}$.

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\tan\frac{x}{2}}\times\frac{d}{dx}\left(\tan\frac{x}{2}\right)$$

$$=\frac{\frac{1}{2}\sec^2\frac{1}{2}}{\tan\frac{x}{2}}=\frac{1}{2}\times\frac{1}{\cos^2\frac{x}{2}}\times\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}$$

$$=\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{1}{\sin x}\operatorname{cosec} x.$$

4·5. Differential coefficients of a^{mx}.

Let $y=a^{mx}$

then $y+\delta y=a^{m+(x+\delta x)}$

or $\delta y=a^{m+(x+\delta x)}-a^{mx}$

$=a^{mx}\,(a^{m\delta x}-1)$

or $$\frac{\delta y}{\delta x}=a^{mx}\times\frac{a^{m\delta x}-1}{\delta x}$$

$$=a^{mx}\times\frac{a^{m\delta x}-1}{m\delta x}\times m.\qquad\text{[Please note this step]}$$

$$\therefore\quad \frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}=ma^{mx}\times\underset{\delta x\to 0}{\text{Lt}}\frac{a^{m\delta x}-1}{m\delta x}$$

But $$\underset{\delta x\to 0}{\text{Lt}}\frac{a^{m\delta x}-1}{m\delta x}=\log a\qquad\left(\because\ \underset{x\to 0}{\text{Lt}}\frac{a^{x}-1}{x}=\log a\right)$$

$$\therefore\quad \frac{dy}{dx}=ma^{mx}\log a.$$

Hence $\frac{d}{dx}(a^{mx})=ma^{mx}\log a$

Or

Differential coefficient of (a^{mx})

$=a^{mx}\times$ *Differential coefficient of* $(mx)\times\log a$.

Example. Differentiate a^{3x} w.r.t. x.

Sol. Let $y=a^{3x}$.

$$\frac{dy}{dx}=a^{3x}\times\frac{d}{dx}(3x)\times\log a$$

$$=3a^{3x}\log a.$$

Cor. *Differential coefficient of* e^{mx} *w.r.t.* $x=me^{mx}$.

In this case $a=e$

$$\therefore\quad \frac{d}{dx}(e^{mx})=me^{mx}\log e=me^{mx}\qquad(\because\ \log e=1)$$

Hence $\frac{d}{dx}(a^{mx})=ma^{mx}\log a$.

4·6. Alternative methods for differentiating a^{mx} and e^{mx} w.r.t. x.

(*i*) Let $y=a^{mx}$

$$\therefore\quad \log y=\log a^{mx}$$
$$=mx\log a.$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=m\log a.$$

$$\therefore\quad \frac{dy}{dx}=my\log a=ma^{mx}\log a.$$

(ii) Let $y=e^{mx}$

$\therefore$ $\log y=\log e^{mx}$

$=mx \log e=mx$ $(\because \log e=1)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx}=m$$

Hence $\frac{dy}{dx}=my=me^{mx}$.

Note 1. The differential co-efficient of $\log y$ w.r.t. x

$$=\frac{1}{y} \cdot \frac{dy}{dx} \text{ and not } \frac{1}{y}.$$

2. The methods explained under Article 4·6 are not *ab-initio* methods.

Solved Examples

Example 1. Differentiate $\log(\log x)$ w.r.t. x.

Sol. Let $y=\log(\log x)$

and $z=\log x$ ...(1)

so that $y=\log z$ ...(2)

Now (1) gives $\frac{dz}{dx}=\frac{1}{x}$

and (2) gives $\frac{dy}{dx}=\frac{1}{z}=\frac{1}{\log x}$

Hence $\frac{dy}{dx}=\frac{dy}{dz} \cdot \frac{dz}{dx}=\frac{1}{\log x} \times \frac{1}{x}=\frac{1}{x \log x}$.

Example 2. Differentiate $e^{\tan x}$ w.r.t. x.

Sol. Let $y=e^{\tan x}$

and let $z=\tan x$ ...(1)

so that $y=e^{z}$ ...(2)

Now from (1) $\frac{dz}{dx}=\sec^2 x$

and from (2) $\frac{dy}{dz}=e^{z}=e^{\tan x}$

$\therefore$ $\frac{dy}{dx}=\frac{dy}{dz} \times \frac{dz}{dx}=e^{\tan x} \times \sec^2 x$

$=\sec^2 x \, e^{\tan x}$.

Alternative Method :

Let $y=e^{\tan x}$

Taking log both the sides, we have

$$\log y=e^{\tan x}=\tan x \log e$$
$$=\tan x.$$

Differentiating this w.r.t. x, we get

$$\frac{1}{y}\ \frac{dy}{dx}=\sec^2 x$$

Hence $\dfrac{dy}{dx}=y \sec^2 x=\sec^2 x\ e^{\tan x}.$

Example 3. If $y=\log(x+\sqrt{1+x^2})$, find $\dfrac{dy}{dx}$.

Sol. Let $z=x+\sqrt{1+x^2}$ ...(1)

so that $y=\log z$ ...(2)

Now
$$\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=\frac{1}{z}\times\frac{d}{dx}\left(x+\sqrt{1+x^2}\right)$$
$$=\frac{1}{x+\sqrt{x^2+1}}\times\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$
$$=\frac{x+\sqrt{1+x^2}}{\sqrt{x^2+1}\,(x+\sqrt{x^2+1})}=\frac{1}{\sqrt{1+x^2}}.$$

Exercise IX

I. Differentiating the following w.r.t. x :

1. $\log\sqrt{\dfrac{x^2-1}{x^2+1}}$.

2. $\log\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}$.

3. $\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$.

4. $\log(\sec x+\tan x)$.

5. $\log\dfrac{1-\cos x}{1+\cos x}$.

6. $\log(x\tan x)$.

7. $\log\tan^{-1}x$.

8. $\log(\log x)$.

9. $\log\left\{e^x\left(\dfrac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$. *(K.U. 1957, 77)*

10. $\log$ than $\dfrac{x}{2}$.

II. Differentiate the following w.r.t. x :

1. xe^x.

2. $e^{ax}+e^{2x}$.

3. $e^{x\tan x}$.

4. $\cos(e^x+e^{-x})$.

5. $\sin e^x$.

6. $a^{\tan(2x+3)}$.

7. $a^{\sin^{-1}x}$.

8. $e^{x\tan^{-1}x}$.

9. $e^{\sqrt{1+x^2}}$.

10. $\log(x^3+e^x)$.

III. 1. If $\log(xy)=x^2+y^2$, show that

$$\frac{dy}{dx}=\frac{y}{x}\cdot\frac{2x^2-1}{1-2y^2}.$$

2. If $y=\log_y x$, show that

$$\frac{dy}{dx}=\frac{1}{x(1+\log y)}$$ *(J. & K. 1950)*

3. If $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots\text{to }\infty}}}$

show that $\frac{dy}{dx}=\frac{1}{x\,(2y-1)}$.

4. If $y=\log_x 10$, show that

$$\frac{dy}{dx}=-\frac{\log 10}{x\,(\log x)^2}.$$

5. If $y=\log x=x-y$, show that

$$\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}.$$

IV. Differentiate the following w.r.t. x :

1. $\tan^{-1}(\log x)$.

2. $\sin(\log\tan x)$ *(K.U. 1954)*

3. $\sin\left(\log\sqrt{\frac{x}{x+1}}\right)$. *(K.U. Nov. 1975)*

4. $\log\sqrt{\frac{1+x+x^2}{1-x+x^2}}$.

5. $\frac{x^2+1}{a^{x^2-3}}$.

6. $\log\frac{a+b\tan x}{a-b\tan x}$.

7. $\log[\log(\log x)]$.

4·7. Hyperbolic Functions

Definition. $\frac{e^x-e^{-x}}{2}$ *is defined as the Hyperbolic sine of x and is written as sinh x.* Similarly, other hyperbolic functions are defined as under :

1. Hyperbolic cosine of $x=\frac{e^x+e^{-x}}{2}$ written as *cosh x*.

2. Hyperbolic tangent of $x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$ written as *tanh x*.

3- Hyperbolic cotangent of $x=\frac{e^x+e^{-x}}{e^x-e^{-x}}$ written as *coth x*.

4. Hyperbolic secant of $x=\dfrac{2}{e^x+e^{-x}}$ written as *sech x*.

5. Hyperbolic cosecant of $x=\dfrac{2}{e^x-e^{-x}}$ written as *cosech x*.

From the above definitions, it is easy to see that :

1. $\tanh x=\dfrac{\sinh x}{\cosh x}$.

2. $\coth x=\dfrac{\cosh x}{\sinh x}$.

3. $\text{sech } x=\dfrac{1}{\cosh x}$.

4. $\text{cosech } x=\dfrac{1}{\sinh x}$.

4·8. Some important relations between Hyperbolic Functions

1. $\cosh^2 x-\sinh^2 x=1$.
2. $1-\tanh^2 x=\text{sech}^2 x$.
3. $\coth^2 x-1=\text{cosech}^2 x$.

Proof. 1. $\cosh^2 x-\sinh^2 x=1$

$$\text{L.H.S.}=(\cosh x+\sinh x)(\cosh x-\sinh x)$$

$$=\left(\frac{e^x+e^{-x}}{2}+\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}\right)$$

$$=e^x\times e^{-x}=e^0=1.$$

Relation (2) can be obtained by dividing relation (1) by $\cosh^2 x$, and relation (3) can be obtained by dividing (1) by $\sinh^2 x$.

4·9. Differentiation of Hyperbolic Functions

(1) Let $\qquad y=\sinh x=\dfrac{e^x-e^{-x}}{2}$

$\therefore \qquad \dfrac{dy}{dx}=\dfrac{e^x+e^{-x}}{2}=\cosh x$

Hence $\qquad \dfrac{d}{dx}(\sinh x)=\cosh x$

(2) Let $\qquad y=\cosh x=\dfrac{e^x+e^{-x}}{2}$

$\therefore \qquad \dfrac{dy}{dx}=\dfrac{e^x+e^{-x}}{2}=\sinh x$

Hence $\qquad \dfrac{d}{dx}(\cosh x)=\sinh x.$

(3) Let $$y=\tanh x=\frac{\sinh x}{\cosh x}$$

$\therefore$ $$\frac{dy}{dx}=\frac{\cosh^2 x-\sinh^2 x}{(\cosh x)^2}$$

$$=\frac{1}{\cosh^2 x}=\text{sech}^2 x.$$

(4) Let $$y=\coth x=\frac{\cosh x}{\sinh x}$$

$\therefore$ $$\frac{dy}{dx}=\frac{\sinh x \,.\, \sinh x-\cosh x \,.\, \cosh x}{(\sinh x)^2}$$

$$=\frac{\cosh^2 x-\sinh^2 x}{\sinh^2 x}=-\frac{1}{\sinh^2 x}$$

$$=-\text{cosech}^2 x.$$

(5) Let $$y=\text{sech } x=\frac{1}{\cosh x}$$

$\therefore$ $$\frac{dy}{dx}=\frac{0\times\cosh x-1\times\sinh x}{(\cosh x)^2}$$

$$=-\frac{\sinh x}{\cosh x}\cdot\frac{1}{\cosh x}$$

$$=-\text{sech } x\tanh x.$$

(6) Let $$y=\text{cosech } x=\frac{1}{\sinh x}.$$

$\therefore$ $$\frac{dy}{dx}=\frac{0\times\sinh x-1\times\cosh x}{(\sinh x)^2}$$

$$=-\frac{\cosh x}{\sinh x}\cdot\frac{1}{\sinh x}$$

$$=-\text{cosech } x\coth x.$$

Note. *The student is advised to mark the difference between the derivatives of circular functions and those of the hyperbolic functions. More or less, they follow the same order, but the last three derivatives in the case of hyperbolic functions are negative, whereas in the case of circular functions, these are alternately positive and negative.*

4·10. Differentiation of Inverse Hyperbolic Functions

(1) Let $y=\sinh^{-1} x$ or $x=\sinh y$

$\therefore$ $$\frac{dx}{dy}=\cosh y$$

Hence $$\frac{dy}{dx}=\frac{1}{\cosh y}$$

$$=\frac{1}{\sqrt{1+\sinh^2 y}}=\frac{1}{\sqrt{1+x^2}}.$$

(2) Let $y=\cosh^{-1} x$

or $x=\cosh y$

$\therefore$ $\dfrac{dx}{dy}=\sinh y$

Hence $\dfrac{dy}{dx}=\dfrac{1}{\sinh y}$

$$=\frac{1}{\sqrt{\cosh^2 y-1}}=\frac{1}{\sqrt{x^2-1}}.$$

(3) Let $y=\tanh^{-1} x$

or $x=\tanh y$

$\therefore$ $\dfrac{dx}{dy}=\operatorname{sech}^2 y$

Hence $\dfrac{dy}{dx}=-\dfrac{1}{\operatorname{sech}^2 y}$

$$=\frac{1}{1-\tanh^2 y}=\frac{1}{1-x^2}.$$

(4) Let $y=\coth^{-1} x$

or $x=\coth y$

$\therefore$ $\dfrac{dx}{dy}=-\operatorname{cosech}^2 y$

Hence $\dfrac{dy}{dx}=-\dfrac{1}{\operatorname{cosech}^2 y}$

$$=-\frac{1}{\coth^2 y-1}=-\frac{1}{x^2-1}.$$

(5) Let $y=\operatorname{sech}^{-1} x$

or $x=\operatorname{sech} y$

$\therefore$ $\dfrac{dx}{dy}=-\operatorname{sech} y \tanh y$

Hence $\dfrac{dy}{dx}=-\dfrac{1}{\operatorname{sech} y \tanh y}=-\dfrac{1}{\operatorname{sech} y\sqrt{1-\operatorname{sech}^2 y}}$

$$=-\frac{1}{x\sqrt{1-x^2}}.$$

(6) Let $y=\operatorname{cosech}^{-1} x$

or $x=\operatorname{cosech} y$

$\therefore$ $\dfrac{dx}{dy}=-\operatorname{cosech} y \coth y$

Hence $\dfrac{dy}{dx}=-\dfrac{1}{\operatorname{cosech} y \coth y}$

$$=-\frac{1}{\text{cosech } y\sqrt{\text{cosech}^2\ y+1}}$$

$$=-\frac{1}{x\sqrt{x^2+1}}.$$

Note. We have *knowingly* avoided writing $\pm$ signs outside the radicals, and the students need not go into such niceties at this stage.

4·11. Logarithmic Differentiation

In order to differentiate a function of the form u^v, where u and v are both variables, it is necessary to take its logarithm and then differentiate. This process is known as *Logarithmic Differentiation,* and is also useful when the function to be differentiated is the product of a number of factors. The following solved examples will illustrate the method :

Example 1. Differentiate $(\tan x)^{\sin x}$ w.r.t. x.

Sol. Let $\qquad y=(\tan x)^{\sin x}$

or $\qquad \log y=\log (\tan x)^{\sin x}$

$$=\sin x \log \tan x.$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\cos x\log\tan x+\sin x\cdot\frac{\sec^2 x}{\tan x}$$

$$=\cos x\log\tan x+\sec x$$

Hence $\qquad \dfrac{dy}{dx}=y\,(\cos x\log\tan x+\sec x)$

$$=(\tan x)^{\sin x}(\cos x\log\tan x+\sec x).$$

Example 2. Differentiate $x^{\tan x}+(\sin x)^{\cos x}$.

Sol. Let $\qquad y=x^{\tan x}+(\sin x)^{\cos x}$

and $\qquad u=x^{\tan x} \qquad \ldots(1)$

$\qquad v=(\sin x)^{\cos x} \qquad \ldots(2)$

so that $\qquad y=u+v$

$\therefore \qquad \dfrac{dy}{dx}=\dfrac{du}{dx}+\dfrac{dv}{dx}.$

From (1) taking logarithm, we obtain,

$$\log u=\tan x\log x$$

$$\therefore \qquad \frac{1}{u} \cdot \frac{du}{dx} = \sec^2 x \log x + \tan x \cdot \frac{1}{x}$$

i.e.
$$\frac{du}{dx} = x^{\tan x}\left(\sec^2 x \log x + \frac{\tan x}{x}\right).$$

From (2) taking logarithm, we obtain,

$$\log v = \cos x \log \sin x$$

$$\therefore \qquad \frac{1}{v} \cdot \frac{dv}{dx} = -\sin x \,.\, \log \sin x + \cos x \,.\, \frac{\cos x}{\sin x}$$

$$= -\sin x \log \sin x + \frac{\cos^2 x}{\sin x}$$

or
$$\frac{dv}{dx} = (\sin x)^{\cos x}\left(-\sin x \log \sin x + \frac{\cos^2 x}{\sin x}\right)$$

Hence,
$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{\tan x}\left(\sec^2 x \log x + \frac{\tan x}{x}\right)$$

$$+ (\sin x)^{\cos x}\left(-\sin x \log \sin x + \frac{\cos^2 x}{\sin x}\right).$$

Example 3. Differentiate $\dfrac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}$.

Sol. Let
$$y = \frac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}$$

Taking logarithms, we obtain

$$\log y = \tfrac{1}{2} \log x + \tfrac{2}{3} \log (1-2x) - \tfrac{3}{4} \log (2-3x)$$
$$- \tfrac{4}{5} \log (3-4x).$$

Differentiating, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x} + \frac{2}{3} \cdot \frac{-2}{1-2x} - \frac{3}{4} \cdot \frac{-3}{2-3x} - \frac{4}{5} \cdot \frac{-4}{3-4x}$$

$$= \frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}$$

$$\therefore \qquad \frac{dy}{dx} = y\left[\frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}\right]$$

$$= \frac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}\left[\frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)}\right.$$
$$\left. + \frac{16}{5(3-4x)}\right].$$

Example 4. If $y = \dfrac{u}{v}$, where u and v are both functions of x, show by logarithmic differentiation that

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Sol. Taking logarithms of both sides, we get

$$\log y = \log \frac{u}{v} = \log u - \log v.$$

Differentiating w.r.t. x, we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} - \frac{1}{v} \cdot \frac{dv}{dx}$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{uv}$$

$$\therefore \qquad \frac{dy}{dx} = y\left(\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{uv}\right)$$

$$= \frac{u}{v}\left(\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{uv}\right)$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Exercise X

(Section A)

Differentiate w.r.t. x :

1. $\sin x \log x \,.\, e^x \,.\, \sqrt{x}$. 2. $\sqrt{x} \,.\, \sqrt{\sin x} \,.\, \sqrt{\log x}$.

3. x^{1+x}. 4. $(\tan x)^{\cot x}$. 5. $(\sin^{-1} x)^x$.

6. $x^{\sin x}$. 7. $(\sin x)^{\log x}$. 8. $\cos (x^x)$. *(K.U. 1977)*

9. a^{b^x}. 10. x^{x^x}. 11. $x^x + x^{1/x}$.

12. $(\sin x)^{\cos x} + (\cos x)^{\sin x}$.

13. $(\sec x)^{\mathrm{cosec}\, x} + (\mathrm{cosec}\, x)^{\sec x}$.

14. $(\tan x)^x + x^{\tan x}$.

15. $\dfrac{2^x \cot x}{\sqrt{x}}$. 16. $\dfrac{x \cos^{-1} x}{\sqrt{1-x^2}}$.

17. If $y = uv$, where u and v are functions of x, show by logarithmic differentiation that $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

18. If $y=x^{x^{x^{\cdots\cdots \text{to } \infty}}}$ prove that

$$x\frac{dy}{dx}=\frac{y}{1-y\log x}.$$

19. If $y=(\cos\theta)^{(\cos\theta)^{(\cos\theta)\cdots\cdots \text{to } \infty}}$, prove that

$$\frac{dy}{d\theta}=\frac{y^2\tan\theta}{y\log\cos\theta-1}.$$

20. If $x^y=e^{x-y}$, prove that $\dfrac{dy}{dx}=\dfrac{\log x}{(1+\log x)^2}$. **(*P.U.*)**

21. If $y=x\log y$, prove that $x\dfrac{dy}{dx}=\dfrac{y^2}{y-x}$. **(*P.U.*)**

22. If $x=\dfrac{e^y-e^{-y}}{2}$, show that $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1+x^2}}$. **(*P.U.*)**

23. If $x=\dfrac{e^y+e^{-y}}{2}$, prove that $\dfrac{dy}{dx}\sqrt{x^2-1}=1$. **(*P.U.*)**

Section (B)

Differentiate the following :

1. $\sinh x+\frac{1}{3}\sinh^3 x$, $\dfrac{\cosh x+\cosh x}{\sinh x+\sinh x}$.
2. $\log\cosh x$, $\log\tanh x$, $a^{\sinh x}$.
3. $\sinh^{-1}\sqrt{x^2-1}$, $\sinh^{-1}(\tan x)$.
4. $\sec^{-1}(\cosh x)$, $\tan^{-1}(\sinh x)$, $\cos^{-1}(\operatorname{sech} x)$.
5. $2\tanh^{-1}(\tan\frac{1}{2}x)$, $\cosh^{-1}(\sec x)$.

5

Successive Differentiation

5·1. Introduction. The derivative $f'(x)$ of a variable function $f(x)$ is itself a function of x. We suppose that it also possesses a derivative which we denote by $f''(x)$ and call the *second derivative* of $f(x)$. The third derivative of $f(x)$ which is the derivative $f''(x)$ is denoted by $f'''(x)$ and so on.

Thus the successive derivatives of $f(x)$ are represented by the symbols

$$f'(x), f''(x), f'''(x), \ldots\ldots\ldots f^n(x)$$

where each term is the derivative of the preceding one.

Alternatively, if $y=f(x)$, then $\dfrac{d^ny}{dx^n}$ denotes the nth derivative of y. Sometimes $y_1, y_2, y_3 \ldots\ldots y_n$ are also used to denote the successive derivatives of y.

Solved Examples

Example 1. Find the third derivative of

$$5x^5+6x^4+3x^2+1.$$

Sol. Let $\quad y=5x^5+6x^4+3x^2+1.$

Denoting the first, the second and the third derivatives of y by y_1, y_2, y_3 respectively, we have

$$y_1=25x^4+24x^3+6x$$

$$y_2=100x^3+72x^2+6$$

and

$$y_3=300x^2+144x.$$

Example 2. If $y=\sin(\sin x)$, prove that

$$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0. \qquad \textit{(K.U. 1977)}$$

Sol. Let $\quad y=\sin(\sin x)$

$$\therefore \quad \frac{dy}{dx}=\cos(\sin x)\,.\,\cos x$$

$$\frac{d^2y}{dx^2}=-\sin(\sin x)\,.\,\cos^2 x-\cos(\sin x)\,.\,\sin x$$

Making substitutions in (1), we get

$$-\sin(\sin x)\,.\cos^2 x-\cos(\sin x)\sin x+\tan x \times\cos(\sin x)\cos x+\sin(\sin x)\cos^2 x=0.$$

Example 3. If $x=f(t)$, $y=\varphi(t)$, prove that

$$\frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dy}\right)^3}$$ *(K.U. 1975)*

Sol. $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$ (Please note this step)

$$=\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\cdot\frac{dt}{dx}$$

$$=\frac{\frac{d^2y}{dt^2}\cdot\frac{dx}{dt}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}\times\frac{1}{\frac{dx}{dt}}$$

$$=\frac{\frac{d^2y}{dt^2}\cdot\frac{dx}{dt}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}.$$

Note. It has been observed that students generally commit mistakes while solving questions of the type of solved example 3. They commit mistakes particularly in respect of the denominator of the above result. They are, therefore, advised to take this example as an article and understand it as thoroughly as possible.

Example 4. If $x=a\,(\theta-\sin\theta)$, $y=a\,(1-\cos\theta)$ find $\frac{d^2y}{dx^2}$.

Sol. Hence $\frac{dx}{d\theta}=a\,(1-\cos\theta)$, $\frac{dy}{d\theta}=a\sin\theta$

$$\therefore\quad \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{a\sin\theta}{a(1-\cos\theta)}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2a\sin^2\frac{\theta}{2}}=\cot\frac{\theta}{2}$$

Again, $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\cot\frac{\theta}{2}\right)$$

$$=\frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\times\frac{d\theta}{dx}$$

$$=-\frac{1}{2}\,.\,\operatorname{cosec}^2\frac{\theta}{2}\times\frac{1}{a(1-\cos\theta)}$$

$$=-\frac{1}{4a}\,.\,\operatorname{cosec}^4\frac{\theta}{2}\cdot$$

Example 5. If $y=\sin(m\sin^{-1}x)$, prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0.$$

Sol. $$y=\sin(m\sin^{-1}x) \qquad \ldots(1)$$

$\therefore$ $$\frac{dy}{dx}=\cos(m\sin^{-1}x)\,.\,\frac{m^2}{\sqrt{1-x^2}}$$

Squaring and multiplying cross-wise, we get

$$(1-x^2)\left(\frac{dy}{dx}\right)=m^2\cos^2(m\sin^{-1}x)$$

$$=m^2[1-\sin^2(m\sin^{-1}x)]$$

$$=m^2(1-y^2) \qquad \text{[from (1)]}$$

Differentiating both sides w.r.t. x, we get

$$(1-x^2)\,.\,2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+(-2x)\left(\frac{dy}{dx}\right)^2=m^2\left(-2y\frac{dy}{dx}\right)$$

or cancelling out $2\frac{dy}{dx}$, we get

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0.$$

Note. It should be noted that the differential coefficient $\left(\frac{dy}{dx}\right)^2$ w.r.t. x is $2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}$ and that of y^2 is $2y\frac{dy}{dx}$.

Exercise XI

Find the

1. Fourth derivative of x^9.
2. Third differential co-efficient of $\frac{1}{x}$.
3. Fifth derivative of $\frac{1}{ax+b}$.
4. Second derivative of $\log(3-x)$.
5. Third derivative of $\tan x+\cot x$.

6. Second differential coefficient of $\{x+\sqrt{x^2-1}\}^n$.

7. If $y=\dfrac{ax+b}{cx+d}$, show that $2y'y'''=3y''^2$.

8. If $y=\dfrac{\log x}{x}$, show that $\dfrac{d^2y}{dx^2}=\dfrac{2\log x-3}{x^3}$.

9. If $y=\log(\sin x)$, show that $y_3=\dfrac{2\cos x}{\sin^3 x}$.

10. If $y=x+\tan x$, prove that

$$\cos^2 x\,\frac{d^2y}{dx^2}-2y+2x=0.$$

11. If $y=A\cos nx+B\sin nx$, then

$$\frac{d^2y}{dx^2}+n^2y=0.$$

12. If $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)+\tan^{-1}\dfrac{2x}{1-x^2}$,

show that $$y_2=-\frac{5x}{(1+x^2)^2}.$$

[**Hint.** Put $x=\tan\theta$, etc.]

13. If $x=(a+bt)\,e^{-nt}$, show that

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+n^2x=0.$$

14. If $x=2\cos t-\cos 2t$,
$y=2\sin t-\sin 2t$, find

$$\frac{d^2y}{dx^2} \text{ at } t=\frac{\pi}{2}.$$

15. If $x=a\cos\theta$, $y=b\sin\theta$, find $\dfrac{d^2y}{dx^2}$.

16. If $x=\sin t$, $y=\sin pt$, prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0.$$

17. (*i*) If $y=(\sin^{-1}x)^2$, prove that

$$(1-x^2)y_2-xy_1=2.$$

(*ii*) If $y=\sin(\sin x)$, prove that

$$y_2+y_1\tan x+y\cos^2 x=0.$$

18. If $ky=\sin(x+y)$ where k is a constant, prove that

$$y_2=-y(1+y_1)^3.$$

19. If $x=\cos\theta$, $y=\sin^3\theta$ show that

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=3\sin^2\theta\,(5\cos^2\theta-1).$$ (*K.U. Nov. 1975*)

5·2. Standard Results

Below we establish a number of standard results which often enable us to find the nth derivatives of various functions. The student is advised to have a clear graph of these results in his own interests.

1. If $y=(ax+b)^m$, to show that

$$y_n=\frac{m\,!}{(m-n)\,!}\,a^n(ax+b)^{m-n}.$$

Here $y=(ax+b)^m$

$\therefore$

$$y_1=ma(ax+b)^{m-1}$$

$$y_2=m(m-1)a^2(ax+b)^{m-2}$$

$$y_3=m(m-1)(m-2)\,a^3(ax+b)^{m-3}.$$

So that in general

$$y_n=m(m-1)(m-2)\ldots\ldots(m-n+1)a^n(ax+b)^{m-n}.$$

If m is +ve integer, this result can be written as

$$y_n=\frac{m\,!}{(m-n)\,!}\,a^n(ax+b)^{m-n}.$$

Cor. 1. This mth derivative of $(ax+b)^m$ is $m\,!\,a^m$ (which is constant).

Cor. 2. If $y=x^n$, then $y_n=n\,!$

2. If $y=\dfrac{1}{ax+b}$, to show that

$$y_n=\frac{(-1)^n\,\underline{|n}\;a^n}{(ax+b)^{n+1}}.$$

Here $y=(ax+b)^{-1}$

$\therefore$

$$y_1=-(ax+b)^{-2}$$

$$=(-1)^1\,\underline{|1}\;a^1(ax+b)^{-(1+1)}$$

$$y_2=2a^2(ax+b)^{-3}$$

$$=(-1)^2\,\underline{|2}\;a^2(ax+b)^{-(1+2)}$$

$$y_3=-6a^3(ax+b)^{-4}$$

$$=(-1)^3\,\underline{|3}\;a^3(ax+b)^{-(1+3)}.$$

Hence, in general

$$y_n=(-1)^n\,\underline{|n}\;a^n(ax+b)^{-(1+n)}$$

$$=\frac{(-1)^n\,\underline{|n}\;a^n}{(ax+b)^{n+1}}.$$

Cor. If $y=\dfrac{1}{x}$, $y_n=\dfrac{(-1)^n\,\underline{|n}}{x^{n+1}}$.

3. If $y=\log\,(ax+b)$, to show that

$$y_n=\frac{(-1)^{n-1}\,\underline{|n-1}\;a^n}{(ax+b)^n}.$$

Here $\quad y=\log (ax+b)$

$\therefore \quad y_1=\dfrac{a}{ax+b}=a(ax+b)^{-1}$

$y_2=-a^2(ax+b)^{-2}$
$\quad =(-1)^1 \,\underline{|1}\, a^2(ax+b)^{-2}$

$y_3=2a^3(ax+b)^{-3}$
$\quad =(-1)^2 \,\underline{|2}\, a^3(ax+b)^{-3}$

....

Hence $\quad y_n=(-1)^{n-1} \,\underline{|n-1}\, a^n(ax+b)^{-n}$

$$=\frac{(-1)^{n-1} \,\underline{|n-1}\, a^n}{(ax+b)^n}.$$

Cor. If $\quad y=\log x,$

then $\quad y_n=\dfrac{(-1)^{n-1} \,\underline{|n-1}}{x^n}.$

4. If $\quad y=a^{mx}$, to show that

$y_n=m^n a^{mx} (\log a)^n.$

Here $\quad y=a^{mx}$

$\therefore \quad y_1=ma^{mx} \log a$

$y_2=m^2a^{mx} (\log a)^2$

$y_3=m^3a^{mx} (\log a)^3$

....

Hence $\quad y_n=m^n a^{mx} (\log a)^n.$

Cor. If $\quad y=e^{mx},$

then $\quad y_n=m^n e^{mx}.$

This result can also be established by putting $a=e$ in 4 above.

5. If $\quad y=\sin (ax+b)$, to show that

$$y_n=a^n \sin \left(ax+b+n\frac{\pi}{2}\right)$$

Here $\quad y=\sin (ax+b)$

$\therefore \quad y_1=a \cos (ax+b)$

$$=a \sin \left(ax+b+\frac{\pi}{2}\right)$$

$$\left[\because \quad \sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta\right]$$

$$y_2=a^2 \cos \left(ax+b+\frac{\pi}{2}\right)$$

$$=a^2 \sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$=a^2 \sin\left(ax+b+2\,\frac{\pi}{2}\right)$$

$$y_3=a^3 \cos\left(ax+b+2\,\frac{\pi}{2}\right)$$

$$=a^3 \sin\left(ax+b+2\,\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$=a^3 \sin\left(ax+b+\frac{3\pi}{2}\right)$$

....

Hence $\quad y_n=a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

Similarly, if $y=\cos(ax+b)$,

then $\quad y_n=a^n \cos\left(ax+b+\frac{n\pi}{2}\right).$

This is left as an exercise for the student.

Cor. If $y=\sin x$, then $y_n=\sin\left(x+\frac{n\pi}{2}\right)$.

6. If $\quad y=e^{ax}\sin(bx+c)$, to show that

$$y_n=(a^2+b^2)^{n/2}\, e^{ax} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Here $\quad y=e^{ax}\sin(bx+c)$

$\therefore \quad y_1=ae^{ax}\sin(bx+c)+be^{ax}\cos(bx+c).$

Put $\quad a=r\cos\theta$ and $b=r\sin\theta$

so that $\quad r=(a^2+b^2)^{1/2} \quad$ and $\quad \theta=\tan^{-1}\frac{b}{a}$

$\therefore \quad y_1=re^{ax}[\sin(bx+c)\cos\theta+\cos(bx+c)\sin\theta]$
$\quad =re^{ax}\sin(bx+c+\theta).$

Similarly, $\quad y_2=r^2\, e^{ax}\sin(bx+c+2\theta)$
$\quad y_3=r^3\, e^{ax}\sin(bx+c+3\theta)$

....

Hence $\quad y_n=r^n\, e^{ax}\sin(bx+c+n\theta)$

$$=(a^2+b^2)^{n/2}\, e^{ax}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Similarly, if $y=e^{ax}\cos(bx+c)$,

then $$y_n=(a^2+b^2)^{n/2}e^{ax}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right).$$

This is left as an exercise for the student.

5·3. Determination of nth Derivatives of Algebraic Rational Functions. Use of Partial Fractions

In order to find the nth derivative of an algebraic rational function, we have to decompose it into partial fractions. The method is explained below by means of a solved example :

Example. Find the nth derivative of $\dfrac{2x-1}{(x-2)(x+1)}$.

Sol. Let $$y=\frac{2x-1}{(x-2)(x+1)}\equiv\frac{A}{x-2}+\frac{B}{x+1}$$

so that $$2x-1\equiv A(x+1)+B(x-2)$$

Putting $x=2$, we get $A=1$

Similarly, putting $x=-1$, we get $B=1$

Hence $$y=\frac{1}{x-2}+\frac{1}{x+1} \qquad \ldots(1)$$

Now if $y=\dfrac{1}{ax+b}$, then $y_n=\dfrac{(-1)^n\,\underline{|n}\;a^n}{(ax+b)^{n+1}}$

Hence (1) gives $$y_n=\frac{(-1)^n\,\underline{|n}}{(x-2)^{n+2}}+\frac{(-1)^n\,\underline{|n}}{(x+1)^{n+1}}.$$

Solved Examples

Example 1. Find the nth differential co-efficient of $\dfrac{1}{3x+2}$.

Sol. Comparing it with $\dfrac{1}{ax+b}$, we find that $a=3$, $b=2$.

$\therefore$ the nth differential co-efficient

$$=\frac{(-1)^n\,\underline{|n}\;a^n}{(ax+b)^{n+1}}=\frac{(-1)^n\,\underline{|n}\;3^n}{(3x+2)^{n+1}}.$$

Alternative Method

Let $$y=\frac{1}{3x+2}=(3x+2)^{-1}$$

$\therefore$
$$y_1=-3(3x+2)^{-2}=(1-1)^1\,\underline{|1}\;3^1(3x+2)^{-(1+1)}$$
$$y_2=18(3x+2)^{-3}=(-1)^2\,\underline{|2}\;3^2(3x+2)^{-(1+2)}$$
$$y_3=-162(3x+2)^{-4}=(-1)^3\,\underline{|3}\;3^3(3x+2)^{-(1+3)}$$

....

Hence $$y_n=(-1)^n\,|\underline{n}\;3^n(3x+2)^{-(1+n)}$$
$$=\frac{(-1)^n\,|\underline{n}\;3^n}{(3x+2)^{n+1}}.$$

Note. Both the methods are permissible, but the latter should be preferred to the former.

Example 2. Find the nth derivative of $\dfrac{1}{(2x+3)^2}$.

Sol. Let $$y=\frac{1}{(2x+3)^2}=(2x+3)^{-2}$$

$\therefore$ $$y_1=-4(2x+3)^{-3}=(-1)^1\,|\underline{2}\;.\;2^1\;.\;(2x+3)^{-(1+2)}$$
$$y_2=24(2x+3)^{-4}=(-1)^2\,|\underline{3}\;.\;2^2\;.\;(2x+3)^{-(2+2)}$$

Hence $$y_n=(-1)^n\,|\underline{n+1}\;.\;2^n\;.\;(2x+3)^{-(2+n)}$$
$$=\frac{(-1)^n\,|\underline{n+1}\;2^n}{(2x+3)^{n+2}}.$$

Example 3. Find the nth differential co-efficient of
$$a^{bx}+e^{ax}+\log(ax+b)+\frac{1}{ax+b}.$$

Sol. Let $y=a^{bx}+e^{ax}+\log(ax+b)+\dfrac{1}{ax+b}$

Now $$y_1=ba^{bx}\log a+e^{ax}+\frac{a}{ax+b}+(-a)(ax+b)^{-2}$$
$$=ba^{bx}\log a+ae^{ax}+a(ax+b)^{-1}$$
$$+(-1)^1\,|\underline{1}\;a^1\,(ax+b)^{-(1+1)}$$
$$y_2=b^2\,a^{bx}(\log a)^2+a^2e^{ax}+(-a^2)(ax+b)^{-2}$$
$$+2a^2(ax+b)^{-3}$$
$$=b^2a^{bx}(\log a)^2+a^2\,e^{ax}+(-1)^2\,|\underline{1}\;a^2\,(ax-b)^{-(1+1)}$$
$$+(-1)^2\,|\underline{2}\;a^2\,(ax+b)^{-(1+2)}$$
$$y_3=b^3\,a^{bx}(\log a)^3+a^3e^{ax}+2a^3(ax+b)^{-3}$$
$$(-6a^3)(ax+b)^{-4}$$
$$=b^3a^{bx}\,(\log a)^3+a^3e^{ax}$$
$$+\frac{(-1)^2\,|\underline{2}\;a^3}{(ax+b)^3}+\frac{(-1)^3\,|\underline{3}\;a^3}{(ax+b)^{3+1}}$$

....

Hence, $$y_n=b^n\,a^{bx}\,(\log a)^n+a^n\,e^{ax}$$
$$+\frac{(-1)^{n-1}\,|\underline{n-1}\;a^n}{(ax+b)^n}+\frac{(-1)^n\,|\underline{n}\;a^n}{(ax+b)^{n+1}}.$$

Example 4. Find the nth differential co-efficient of

$\sin x \cos 2x$. *(P.U. 1957)*

Sol. Let $y = \sin x \cos 2x$

$$= \tfrac{1}{2}(2 \cos 2x \sin x)$$

$$= \tfrac{1}{2}(\sin 3x - \sin x).$$

Now each is of the form $\sin ax$ whose nth diff. co-efficient is $a^n \sin\left(ax + \dfrac{n\pi}{2}\right)$.

$\therefore$ Putting $a=3$, $a=1$ respectively, we get

$$y_n = \frac{1}{2}\left[3^n \sin\left(3x + \frac{n\pi}{2}\right) - \sin\left(x + \frac{n\pi}{2}\right)\right].$$

Example 5. If $y = \cos^3 x$, find y_n. *(K.U. 1976)*

Sol. Now $y = \cos^3 x = \tfrac{1}{4}(4 \cos^3 x)$

$$= \tfrac{1}{4}(3 \cos x + \cos 3x)$$

$[\because \cos 2A = 4 \cos^3 A - 3 \cos A]$

Now each is of the form $\cos ax$ whose nth diff. co-efficient is $a^n \cos\left(ax + \dfrac{n\pi}{2}\right)$.

Putting $a=1$ and $a=3$ respectively, we have

$$y_n = \frac{1}{4}\left[3 \cos\left(x + \frac{n\pi}{2}\right) + 3^n \cos\left(3x + \frac{n\pi}{2}\right)\right].$$

Example 6. Find y_n if $y = e^{2x} \sin 3x \sin 4x$.

Sol. Let $y = e^{2x} \sin 3x \sin 4x$

$$= \tfrac{1}{2}e^{2x}(2 \sin 4x \sin 3x)$$

$$= \tfrac{1}{2}e^{2x}(\cos x - \cos 7x)$$

$$= \tfrac{1}{2}[e^{2x} \cos x - e^{2x} \cos 7x].$$

Now the term in each part is of the standard form

"$e^{ax} \cos bx$".

$$\therefore \quad y_n = \tfrac{1}{2}(2^2 + 1^2)^{n/2} e^{2x} \cos (x + n \tan^{-1} \tfrac{1}{2}) + (2^2 + 7^2)^{n/2} e^{2x} \cos (7x + n \tan^{-1} \tfrac{7}{2}).$$

Exercise XII

Find the nth derivatives of :

1. $\dfrac{1}{1+x}$.
2. $\dfrac{1}{a-x}$.
3. $\dfrac{1}{a-bx}$.
4. $\dfrac{x+2}{3x+7}$. *(K.U. 1976)*
5. $\dfrac{1}{(2x+3)^2}$.
6. $\sin x \sin 2x$.
7. $a \sin^2 x + b \cos^2 x$.
8. $e^{2x} + a^{5x}$.

9. $\cos x \cos 2x \cos 3x$. **10.** $\sin^2 x$.

10 (*a*). If $y=\dfrac{1}{x^2+6x+8}$, show that

$$y_n=\frac{(-1)^n \lfloor n}{2}\left[\frac{1}{(x+2)^{n+1}}-\frac{1}{(x+4)^{n+2}}\right].$$ (*K.U. 1975*)

11. $\cos^4 x$. **12.** $e^x \cos x \cos 2x$.

13. $e^x (\cos x+\sin x)$. **14.** $e^{x \cos a} \cos (x \sin a)$.

15. $\sin^3 x$. **16.** $e^x \sin^2 x$.

17. $e^x \cos^3 x$. **18.** $e^x \sin \sqrt{3}x$. (*P.U. 1958, 55*)

19. $e^{5x} \sin 3x \cos 2x$. **20.** $e^{3x} \sin 4x$.

5·4. Leibnitz's Theorem for the nth differential coefficient of the product of two functions of x.

Statement. *If $y=uv$, where u and v are the functions of x having derivatives of any desired order, then*

$$y_n={}^nC_0\, u_n v+{}^nC_1\, u_{n-1}\, v_1+{}^nC_2\, u_{n-2}\, v_2+\ldots\ldots$$
$$\ldots\ldots{}^nC_r u_{n-r}\, v_r+\ldots\ldots\; {}^nC_n u v_n$$

where u_r and v_r denote the nth differential coefficient of u and v w.r.t. x respectively.

We shall prove this *important* theorem by the method known as *Mathematical Induction.*

Let us suppose that the theorem is true for any *particular value* of *n*. This means that this equation is true for this particular value of *n*. Differentiating this equation once again w.r.t. *x*, we get

$$y_{n+1}={}^nC_0(u_{n+1}v+u_n v_1)+{}^nC_1(n_n v_1+u_{n-1}v_2)+{}^nC_2(u_{n-1}v_2+u_{n-2}v_3)+\ldots\ldots+{}^nC_r(u_{n-r+1}v_r+u_{n-r+1})+\ldots\ldots$$
$$\ldots\ldots+{}^nC_n(u_1 v_n+uv_{n+1})$$
$$={}^nC_0 u_{n+1}v+({}^nC_0+{}^nC_1)u_n v_1+({}^nC_1+{}^nC_2)u_{n-1}v_2+\ldots\ldots+({}^nC_{r-1}+{}^nC_r)u_{n-r+1}v_r+\ldots\ldots+{}^nC_n uv_{n+1} \qquad \ldots(1)$$

Now ${}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$

$\therefore$ Putting $r=1, 2$, we get

$${}^nC_0+{}^nC_1={}^{n+1}C_1 \text{ and } {}^nC_1+{}^nC_2={}^{n+1}C_2$$

Also $${}^nC_0=1={}^{n+1}C_0$$

and $${}^nC_n=1={}^{n+1}C_{n+1}$$

Hence (1) becomes

$$y_{n+1}={}^{n+1}C_0 u_{n+1}v+{}^{n+1}C_1 u_n v_1+{}^{n+1}C_2 u_{n-1}v_2+\ldots\ldots$$
$$+{}^{n+1}C_r u_{n-r+1}v_r+\ldots\ldots+{}^{n+1}C_{n+1}uv_{n+1}.$$

This shows that the theorem is true for $n+1$ as well. Hence, if the formula is true for any particular value of n, it is *equally* true for $n+1$ as well.

Again, by actual differentiation,

if $$y=uv$$

then $$y_1=u_1v+uv_1$$

and
$$\begin{aligned}y_2&=u_2v+u_1v_1+u_1v_1+uv_2\\&=u_2v+2u_1v_1+uv_2\\&={}^2C_0uv_2+{}^2C_1u_1v_1+{}^2C_2uv_2\end{aligned}$$

This again shows that the theorem is true for $n=2$. Hence it is true for

$n=2+1$, *i.e.* for $n=3$ and, therefore, for

$n=3+1$, *i.e.* for $n=4$ and so on.

Hence we conclude that the theorem is true for all *positive integral* values of n.

Solved Examples

Example 1 Find the nth derivative of x^2e^x.

Sol. Let $y=x^2e^x$, and $u=e^x$

and $v=x^2$, so that

$u_n=e^x$, $u_{n-1}=e^x$, $u_{n-2}=e^x$, etc. and $v_1=2x$, $v_2=2$

$\therefore$ $y_n={}^nC_0u_nv+{}^nC_1u_{n-1}v_1+{}^nC_2u_{n-2}v_2+\ldots\ldots$ gives

$$\begin{aligned}y_n&={}^nC_0\,e^x.x^2+{}^nC_1.e^x.2x+{}^nC_2.e^x.2\\&=x^2e^x+n.e^x.2x+(n-1)e^x\\&=e^x(x^2+2nx+n^2-n).\end{aligned}$$

Note. In the above example, we have put $e^x=u$ because we can find the nth derivative of e^x; we cannot do so from x^2, because x^2 can give us only two derivatives, *i.e.*, $2x$ and 2. Hence we have put $x^2=v$. The student is, therefore, advised to take that function as u which can *easily* give him the nth derivative. He is at liberty to make a *free* selection in this behalf if both u and v can give nth derivatives with equal case.

Example 2. If $y=e^{a\sin^{-1}x}$, prove that

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0.$$

(P.U. 1955)

Sol. We have $y_1=e^{a\sin^{-1}x}.\dfrac{a}{\sqrt{1-x^2}}=\dfrac{ay}{\sqrt{1-x^2}}$

or $$y_1^2(1-x^2)=a^2y^2$$

Differentiating again, we get

$$2y_1y_2(1-x^2)-2xy_1^2=2a^2yy_1$$

or $$y_2(1-x^2)-xy_1-a^2y=0$$

Differentiating this equation n times with the help of Leibnitz's Theorem, we get

$$y_{n+2}(1-x^2)+{}^nC_1y_{n+1}(-2x)+{}^nC_2y_n(-2)$$
$$xy_{n+1}-{}^nC_1.1.y_n-a^2y_n=0$$

which on simplification, gives

$$(1-x^2)y_{n+1}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0.$$

Example 3. Find $(y_n)_0$, if $y=\cos(m\sin^{-1}x)$.

Sol. Here $y=\cos(m\sin^{-1}x)$ $\quad\therefore\quad (y)_0=1$

$$y_1=-\sin(m\sin^{-1}x).\frac{m}{\sqrt{1-x^2}} \qquad \therefore\quad (y_1)_0=0$$

or $\quad y_1^2(1-x^2)=m^2\sin^2(m\sin^{-1}x)$
$$=m^2[1-\cos^2(m\sin^{-1}x)]=m^2(1-y^2).$$

Differentiating again, we get

$$2y_1y_2(1-x^2)-2xy_1^2=-2m^2yy_1$$

or $\quad y_2(1-x^2)-xy_1+m^2y=0 \qquad \ldots(1)$

$\therefore \quad (y_2)_0+m^2(y)_0=0$

or $\quad (y_2)_0=-m^2(y)_0=-m^2 \qquad [\because\ (y)_0=1]$

Differentiating (1) n times, we get

$$y_{n+2}(1-x^2)+{}^nC_1.y_{n+1}.(-2x){}^nC_2y^n.(-2)-xy_{n+1}-{}^nC_1.1.y_n+m^2y_n=0$$

Putting $x=0$, we get

$$(y_{n+2})_0=(n^2-m^2)(y_n)_0$$

or $\quad (y_n)_0=\{(n-2)^2-m^2\}(y_{n-2}).$

Let n be odd.

$\therefore$ Putting $n=3, 5, 7$, etc., we get

$$(y_3)_0=(1^2-m^2)(y_1)_0=0 \qquad \because\ (y_1)_0=0$$
$$(y_5)_0=(3^2-m^2)(y_3)_0=0 \qquad \because\ (y_3)_0=0$$
$$(y_7)_0=(5^2-m^2)(y_5)_0=0 \qquad \because\ (y_5)_0=0$$

Hence $(y_n)=0$ when n is odd.

Also putting $n=4, 6, 8$, etc., we get

$$(y_4)_0=(2^2-m^2)(y_2)_0$$
$$=(2^2-m^2)(-m)^2. \qquad \because\ (y_2)_0=-m^2$$
$$(y_6)_0=(4^2-m^2)(y_4)_0$$
$$=(4^2-m^2)(2^2-m^2)(-m^2), \text{ etc.}$$

Hence when n is even, we have

$$(y_n)_0=\{(n-2)^2-m^2\}(y_{n-2})_0$$
$$=\{(n-2)^2-m^2\}\{(n-4)^2-m^2\}(y_{n-4})_0$$
$$=\{(n-2)^2-m^2\}\{(n-4)^2-m^2\}\ldots\ldots$$
$$\ldots\ldots(4^2-m^2)(2^2-m^2)(-m)^2.$$

Exercise XIII

Find the nth differential coefficient of :

1. $x^3 \cos x$. **2.** $x^2 \log x$.

3 $e^x \log x$. *(K U. 1976 ; P.U. 1956)*

4. $x^2 a^x$. **5.** $x^2 e^{ax}$.

6. $x^n e^x$. **7.** $x e^{ax} \sin bx$.

8. If $y = a \cos (\log x) + b \sin (\log x)$, show that

(*i*) $x^2 y_2 + x y_1 + y = 0$.

(*ii*) $x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1)\, y_n = 0$.

9. If $y = e^{-x} \cos x$, show that $y_4 + 4y = 0$.

10. If $y = \sin^{-1} x$, prove that

$$(1-x^2) \frac{d^2y}{dx^2} = x \frac{dy}{dx}.$$

11. If $y = e^{a \sin^{-1} x}$, show that

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+a^2) y_n = 0.$$

Deduce the value of $(y_n)_0$.

12. If $y = (\sin^{-1} x)^2$, prove that

(*i*) $(1-x)^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} - 2 = 0,$

and (*ii*) $(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0.$

13. If $y = (x^2-1)^n$, prove that

$(x^2-1) y_{n+2} + 2x y_{n+1} - n(n+1) y_n = 0.$ *(K.U. 1975)*

14. If $y = [x + \sqrt{1+x^2}]^m$, show that

(*i*) $y_2(1+x^2) + x y_1 - m^2 y = 0,$

and (*ii*) $y_{n+2}(1+x^2) + (2n+1) x y_{n+1} + (n^2-m^2) y_n = 0.$ *(K.U. 1976)*

Also calculate $(y_n)_0$.

15. If $y = \sin (m \sin^{-1} x)$, prove that

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2-m^2) y_n = 0.$$

16. If $y^{1/m} + y^{-1/m} = 2x$, prove that

$(x^2-1) y_{n+2} + (2n+1) x y_{n+1} + (n^2-m^2) y_n = 0.$ *(K.U. 1977)*

17. If $y = e^{ax} X$, where X is a function of x, show that

$$y_n = e^{ax}(X_n + {}^nC_1 a X_{n-1} + {}^nC_2 a^2 X_{n-2} + \ldots\ldots)$$

where X_1, X_2,......etc., denote the successive derivates of X with respect to x.

18. If $u = \sin nx + \cos nx$, show that

$$u_r = n^r \{1 + (-1)^r \sin 2nx\}^{1/2}$$

where u_r denotes the rth derivative of u with respect to x.

(K.U. 1961, Supp.)

6

Derivative as a Rate Measurer

6·1. In differential calculus, we are always concerned with the variation of related quantities. This means that if one quantity changes, we wish to know *the rate of change* of any other related quantity. This is explained below by means of some examples :

Let $y=f(x)$ be any function of x. If h be the increase in the value of x, then $f(x+h)-f(x)$ will be the increase in the value of $f(x)$, and the ratio $\dfrac{f(x+h)-f(x)}{h}$ represents the *average* rate of increase in $f(x)$ as x increases from x to $x+h$. If h be taken smaller, this ratio will measure more and more approximately the rate of increase of $f(x)$ relative to x at the particular value of x under consideration.

Therefore, when $h \to 0$,

$$\text{Limit } \frac{f(x+h)-f(x)}{h}=f'(x)$$

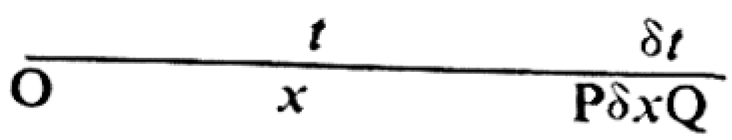
measures the rate of change of $f(x)$ relative to x.

Hence $\dfrac{dy}{dx}$ *measures the rate of change of y with respect to x.*

6·2. Rate of change with respect to time

If a quantity x be a function of the time t, then $\dfrac{dx}{dt}$ measures the rate of change of x with respect to the time t.

Illustrations. Let the position of a moving point in a straight line at time t be fixed by measuring its distance x from the origin O in the straight line. Let this distance be $x+\delta x$ after time $t+\delta t$. Then the total space described in δt time is δx, and, therefore, the average



speed is equal to $\dfrac{\delta x}{\delta t}$. Now as $\delta t \to 0$, the rate at which the space is described at time t is represented by $\dfrac{dx}{dt}$. This is known as the *velocity* at time t and is *generally* denoted by v.

Similarly, the rate of change of velocity at time t will be represented by $\frac{dv}{dt}$. This is known as *acceleration*, and *generally* denoted by f.

Solved Examples

Example 1. Find the rate of change of the volume of a circular cylinder of radius r and height h when the radius varies.

Sol. Let v be the volume, then $v=\pi r^2 h$

$\therefore$ Rate of change of v w.r.t. r is

$$\frac{dv}{dr}=2\pi rh.$$

Example 2. A particle moves in a straight line and its distance S in feet from a fixed point O is given by $S=t(t-1)^2$. Find its velocity and acceleration on each occasion when it passes through O.

Sol. When the particle passes through O, $S=0$. Naturally $t(t-1)^2=0$ which gives $t=0$ and $t=1$. As such, we have to find velocity and acceleration when $t=0$, and $t=1$

Now $$S=t^3-2t^2+t$$

$\therefore$ $$v=\frac{ds}{dt}=3t^2-ut+1$$

and $$f=\frac{dv}{dt}=6t-4$$

Therefore, (*i*) when $t=0$, $v=1'$, and $f=-4'/\text{sec}^2$

and (*ii*) when $t=1$, $v=0$, and $f=2'/\text{sec}^2$.

Note. Acceleration is denoted either by $\frac{dv}{dt}$ or by $\frac{d^2s}{dt^2}$ or by $v\frac{dv}{ds}$.

This is so because $$\frac{dv}{dt}=\frac{d}{dt}\left(\frac{ds}{dt}\right)=\frac{d^2s}{dt^2}$$

Again, $$\frac{dv}{dt}=\frac{dv}{ds}\cdot\frac{ds}{dt}=\frac{dv}{ds}\cdot v=v\frac{dv}{ds}$$

Hence $$f=\frac{dv}{dt}=v\frac{dv}{ds}=\frac{d^2s}{dt^2}.$$

Example 3. A particle moves along a straight line according to the law

$$v^2=2(\sin x-x\cos x)$$

where v is the velocity and x the distance described. Find its acceleration.

Sol. $v^2=2(\sin x-x\cos x)$.

Differentiating w.r.t. x, we get

$$2v\frac{dv}{dx}=2(\cos x-1\,.\,\cos x+x\sin x)$$

$$\therefore \qquad f=v\frac{dv}{dx}=x\sin x.$$

Exercise XIV

1. Find the rate at which the following vary with respect to a change in radius :

(*i*) The area of a circle of radius r,

(*ii*) the total surface of a cylinder of radius r and height h,

(*iii*) the volume of a sphere of radius r.

2. A point moves so that its distance S from a fixed point O at time t is expressed by

$$S=ae^{-kt}\sin(wt+a).$$

Find the velocity of the point at any time t.

3. If $x^2+y^2=25$, find the rate of increase of y at the point $x=3$; also at the point $x=-4$.

4. The radius of a circle is increasing uniformly at the rate of 3" per second. At what rate is the area increasing when the radius is 1 ft. ? (*K.U. 1975*)

5. If a body is moving in a straight line and its distance in feet from a given point in the line after t seconds is given by

$$S=5+2t+4t^3.$$

(*K.U. Old Course, November, 1976*)

Find (*i*) the speed at the end of $2\frac{3}{4}$ seconds,

(*ii*) the acceleration at the end of $3\frac{1}{2}$ seconds,

and (*iii*) the average speed during the 4th second.

6. A particle moves along a straight line, the law of motion being

$$S=A\cos(nt+k)$$

show that the acceleration is directed to two origin and varies as the distance.

7. A point moves in a straight line so that its distance S from a fixed point at any time t is proportional to t^n. If v be the velocity and f the acceleration of any time t, show that

$$v^2=\frac{nsf}{n-1}.$$

8. A particle moves along a straight line according to the law

$$S^2 = at^2 + 2bt + c.$$

Prove that the acceleration varies as $\dfrac{1}{S^3}$.

(*K.U. November, 1975*)

9. The speed of a particle moving along the axis of x is given by

$$v^2 = 4x - x^3.$$

If f is its acceleration, show that

$$27v^4 = 8(2-f)(4+f)^2.$$

10. A triangle has two of its vertices at $(-a, 0)$ and $(a, 0)$ and the third (x, y) moves along the line $y = mx$. If A be its area, show that

$$\frac{dA}{dx} = ma. \qquad \text{(K.U. November, 1976)}$$

(**Hint.** $A = \frac{1}{2}.2amy = max$ etc.)

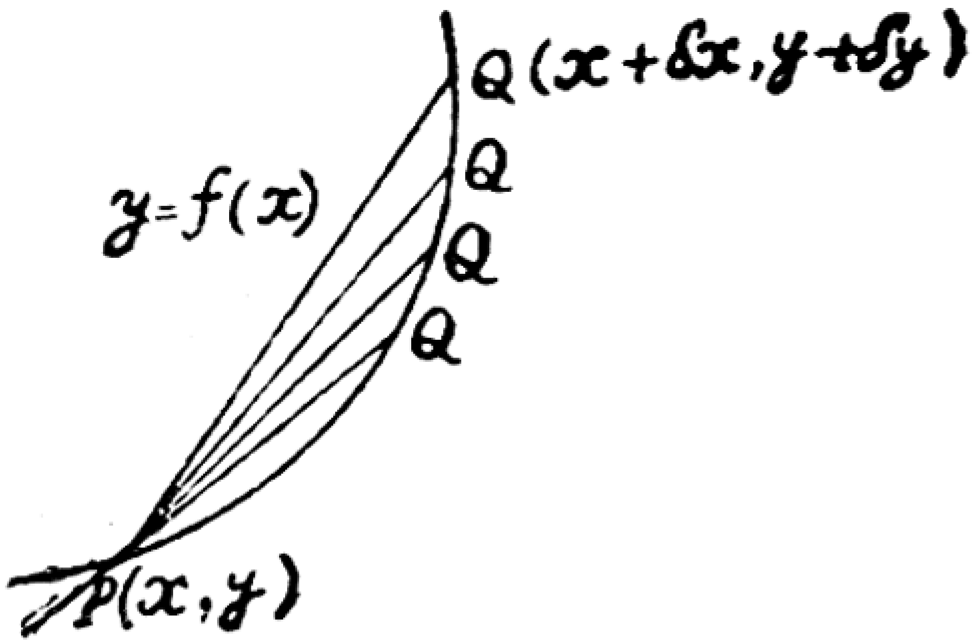
11. The diameter of an expanding smoke ring at time t is proportional to t^2. If the diameter is 6 cm. after 6 seconds, at what rate is it then changing? (*K.U. 1977*)

7

Tangents and Normals

7·1. Definition

Let P(x, y) and Q($x+\delta x$, $y+\delta y$) be any two neighbouring points on the curve $y=f(x)$, then *the limiting position of the chord PQ as Q tends to P along the curve, is called the tangent to the curve at P*.



7·2. Equation of the tangent to the curve y=f(x) at P(x, y)

Let P(x, y) and Q($x+\delta x$, $y+\delta y$) be any two neighbouring points on the curve $y=f(x)$. Then, as we know from Co-ordinate Geometry, the equation of the chord PQ is

$$Y-y=\frac{y+\delta y-y}{x+\delta x-x}\,(X-x)$$

where X and Y are the current co-ordinates.

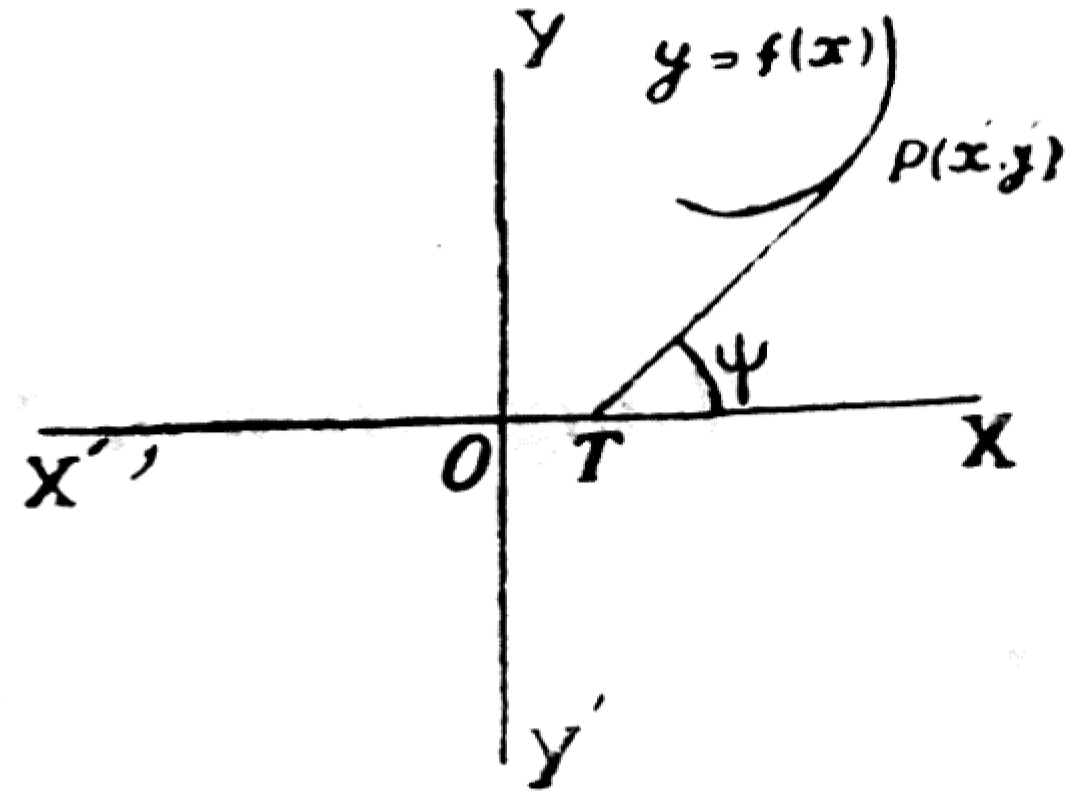
or $$Y-y=\frac{\delta y}{\delta x}\,(X-x).$$

Now that the tangent at P(x, y) is the *limiting position* of the chord PQ as Q →P, proceeding to the limits, as $\delta x \to 0$, we get the required equation as

$$Y-y=\frac{dy}{dx}\,(X-x)\ldots\ldots \qquad \left(\because \ \underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{\delta y}{\delta x}=\frac{dy}{dx}\right)$$

7·3.* Geometrical Meaning of $\frac{dy}{dx}$

In Article 2·5, it has been shown *that the values of the differential co-efficient of a function $f(x)$ at some point is the slope of the tangent to the graph of the function at that point.*



Thus if the tangent PT to the curve $y=f(x)$ at $P(x', y')$ makes an angle ψ with the axis of x, then the slope of this tangent, *i.e.* $\tan\psi$ is equal to the value of the differential co-efficient of the function $y=f(x)$ at the point $P(x', y')$. This can be written as

$$\tan\psi=\left(\frac{dy}{dx}\right)_{x',\, y'}.$$

7·4. Equation of the Normal to the curve y=f(x) at P(x, y)

Definition. *Normal to a curve at any point is a straight line through that point perpendicular to the tangent at that point.*

Hence the slope of the tangent is $\frac{dy}{dx}$, that of the normal will be $-\frac{dx}{dy}$. Hence the equation of the normal at $P(x, y)$ is

$$Y-y=-\left(\frac{dx}{dy}\right)(X-x)$$

or
$$(Y-y)\frac{dy}{dx}+(X-x)=0.$$

Illustrations :

1. Find the equations of tangent and normal to the curve $x^2+y^2=13$ at (3, 2).

Sol. $x^2+y^2=13$

Differentiating this w.r.t. x, we get

$$2x+2y\frac{dy}{dx}=0$$

*For detail, please refer to Article 2·5.

or $$\frac{dy}{dx} = -\frac{x}{y}$$

$\therefore$ Slope of the tangent at $(3, 2)$

$$= \left(\frac{dy}{dx}\right)_{(3,\,2)} = -\frac{3}{2}$$

Hence the equation of the tangent is

$$y - 2 = -\tfrac{3}{2}(x - 3)$$

or $$3x + 2y = 13.$$

Next, slope of the normal $= \frac{2}{3}$

$\therefore$ The equation of the normal is

$$y - 2 = \tfrac{2}{3}(x - 3)$$

or $$2x - 3y = 0.$$

2. Find the equation of the tangent to the curve $y^2 = 4ax$ at any point.

Sol. Let the point be $P(x, y)$.

Differentiating the equation of the curve w.r.t. x, we get

$$2x\frac{dy}{dx} = 4a$$

or $$\frac{dy}{dx} = \frac{2a}{x}.$$

$\therefore$ The slope of the required tangent at $P(x, y)$

$$= \left(\frac{dy}{dx}\right)_{x,\,y} = \frac{2a}{y}.$$

Hence the required equation is

$$Y - y = \frac{2a}{y}(X - x)$$

or $$yY - y^2 = 2aX - 2ax$$

or $$yY = 2aX - 2ax + y^2$$

or $$= 2aX - 2ax + 4ax \qquad (\because y^2 = 4ax)$$

$$= 2a(X + x).$$

An Important Note

In illustration (1) we have not taken capital letters as current co-ordinates because the point of contact is given there in clear terms. In the second illustration, on the other hand, we have taken (x, y) as the point of contact which was not given, and to avoid confusion, we have taken current co-ordinates in capital letters in it. Thus, *current co-ordinates should be taken in capital letters where the point of contact is not given, and where we take it to be (x, y).*

7·5. Equations of tangent and normal when the Parametric Equations of the curve are given

Suppose the equations of the curve are :

$$x=f(t) \qquad \ldots(i)$$

$$x=\varphi(t) \qquad \ldots(ii)$$

Now from (i), we have $\dfrac{dx}{dt}=f'(t)$

and from (ii) $\qquad \dfrac{dy}{dt}=\varphi'(t)$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy}{dt}\bigg/\frac{dx}{dt}=\frac{\varphi'(t)}{f'(t)}.$$

Hence the equation of the tangent at

$$[x=f(t),\ y=\varphi(t)] \quad \text{or} \quad \text{at "}t\text{" is}$$

$$y-\varphi(t)=\frac{\varphi'(t)}{f'(t)}[x-f(t)]$$

Again, the slope of the normal

$$=-\frac{1}{\dfrac{dy}{dx}}=-\frac{f'(t)}{\varphi'(t)}.$$

Hence the equation of the normal is

$$y-\varphi(t)=-\frac{f'(t)}{\varphi'(t)}[x-f(t)]$$

or $\qquad [x-f(t)]f'(t)+[y-\varphi(t)]\varphi'(t)=0$

Note. As is obvious above, we can easily avoid taking current co-ordinates in capital letters in case the equation of a curve is given in parametric form.

Illustration. Find the equations of tangent and normal at the point "θ" to the ellipse

$$x=a\cos\theta,\ y=b\sin\theta. \qquad (P.U.\ 1965)$$

Sol. $\qquad x=a\cos\theta$

$$\therefore \qquad \frac{dx}{d\theta}=-a\sin\theta$$

and $\qquad y=b\sin\theta$

$$\therefore \qquad \frac{dy}{d\theta}=b\cos\theta.$$

Now $\qquad \dfrac{dy}{dx}=\dfrac{dy}{d\theta}\bigg/\dfrac{dx}{d\theta}=-\dfrac{b\cos\theta}{a\sin\theta}.$

Hence the equation of tangent at $(a \cos \theta, b \sin \theta)$ is

$$y-b \sin \theta=-\frac{b \cos \theta}{a \sin \theta}(x-a \cos \theta)$$

or $$bx \cos \theta+ay \sin \theta=ab (\cos^2 \theta+\sin^2 \theta)=ab$$

or $$\frac{x \cos \theta}{a}+\frac{y \sin \theta}{b}=1.$$

Again, the slope of the normal

$$=\frac{a \sin \theta}{b \cos \theta}$$

Hence the equation of the normal at $(a \cos \theta, b \sin \theta)$

$$y-b \sin \theta=\frac{a \sin \theta}{b \cos \theta}(x-a \cos \theta)$$

or $$ax \sin \theta-by \cos \theta=(a^2-b^2) \sin \theta \cos \theta$$

or $$ax \sec \theta-by \operatorname{cosec} \theta=(a^2-b^2).$$

Solved Examples

Example 1. Find the equation of the tangent at any point to the curve $y=f(x)$. *(K U. 1950)*

Sol. Let (x, y) be the point of contact. Now the equation of the curve is

$$y=f(x).$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x, y}=f'(x).$$

Hence the equation of the tangent at (x, y) is

$$Y-y=f'(x)[X-x].$$

Note. *The above example is extremely important and can be taken as an article.*

Example 2. Find the equation of the tangent and the normal at the point "t" to the curve whose equations are

$$x=a \cos^3 t, \qquad y=a \sin^3 t.$$

(K.U. 1964, Nov., 1975, 77)

Sol. From $x=a \cos^3 t$, we have

$$\frac{dx}{dt}=-3a \cos^2 t \sin t$$

and from $y=a \sin^3 t$,

$$\frac{dy}{dt}=3a \sin^2 t \cos t$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{dt} \Big/ \frac{dx}{dt}$$

$$=-\frac{3a \sin^2 t \cos t}{3a \cos^2 t \sin t}=-\frac{\sin t}{\cos t}.$$

Hence the equations of the tangent at $(a \cos^3 t, a \sin^3 t)$ is

$$y-a \sin^3 t=-\frac{\sin t}{\cos t}(x-a \cos^3 t)$$

or $\quad y \cos t-a \sin^3 t \cos t=-x \sin t+a \cos^3 t \sin t$

or $\quad x \sin t+y \cos t=a \sin t \cos t$

or $\quad x \sec t+y \operatorname{cosec} t=a.$

Again, the slope of the normal

$$=-\frac{1}{\frac{dy}{dx}}=\frac{\cos t}{\sin t}.$$

Hence the equation of the normal at $(a \cos^3 t, a \sin^3 t)$ is

$$y-a \sin^3 t=\frac{\cos t}{\sin t}(x-a \cos^3 t)$$

or $\quad y \sin t-a \sin^4 t=x \cos t-a \cos^4 t$

or $\quad x \cos t-y \sin t=a(\cos^4 t-\sin^4 t)$

$$=a(\cos^2 t-\sin^2 t)\times(\cos^2 t+\sin^2 t)$$

or $\quad x \cos t-y \sin t=a \cos 2t.$

Example 3. Obtain the equations of the normals to the curve $(y-3)(y-6)=x^2$ at the point, where $y=7$. *(K.U. Nov. 1976)*

Sol. The equation of the curve when simplified is

$$y^2-9y+18=x^2$$

Differentiating this with respect to x, we get

$$2y\frac{dy}{dx}-9\frac{dy}{dx}=2x$$

or $$\frac{dy}{dx}=\frac{2x}{2y-9}.$$

Now $y=7$, therefore, substituting this in the equation of the curve, wet get, $x=\pm 2$.

$\therefore$ The points at which we have to find the eqations of the normals are $(2, 7)$ and $(-2, 7)$.

Now $\left(\frac{dy}{dx}\right)_{(2,\ 7)}=-\frac{4}{5}$ and $\left(\frac{dy}{dx}\right)_{(-2,\ 7)}=-\frac{4}{5}$

$\therefore$ Slopes of the normals are $-\frac{5}{4}$ and $\frac{5}{4}$.

Hence the equations of the normals are

$$y-7=-\tfrac{5}{4}(x-2) \quad \text{or} \quad 5x+4y=38$$

and $$y-7=\tfrac{5}{4}(x+2) \quad \text{or} \quad 5x-4y+38=0.$$

Example 4. Find the points on the curve

$$y=3x^2-6x+5$$

where the tangents to the curve are parallel to the axis of x.

Sol. The slope of the tangent which is parallel to the x-axis is zero.

Differentiating the equation of the curve w.r.t. x, we get

$$\frac{dy}{dx}=6x-6.$$

As the tangent is parallel to the axis of x, we have

$$\frac{dy}{dx}=0 \quad \text{or} \quad 6x-6=0$$

which gives $x=1$.

Substituting this value of x in the equation of the curve, we get $y=2$.

Hence the required point is (1, 2).

Example 5. Find the point on the curve

$$x^{2/3}+y^{2/3}=a^{2/3}$$

at which the normal makes an angle θ with x-axis.

Sol. Equation of the curve is

$$x^{2/3}+y^{2/3}=a^{2/3} \qquad ...(1)$$

Differentiating w.r.t. x, we get

$$\tfrac{2}{3}x^{-1/3}+\tfrac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

$$\therefore \qquad \frac{dy}{dx}=-\frac{x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}$$

Slope of the normal

$$=-\frac{dy}{dx}=\frac{x^{1/3}}{y^{1/3}}$$

By the given condition,

$$\frac{x^{1/3}}{y^{1/3}}=\tan\theta \qquad ...(2)$$

$$\therefore \qquad \frac{x^{2/3}}{\sin^2\theta}=\frac{y^{2/3}}{\cos^2\theta}=\frac{x^{2/3}+y^{2/3}}{\sin^2\theta+\cos^2\theta}=a^{2/3} \qquad \text{by (1)}$$

$$\therefore \qquad x^{2/3}=a^{2/3}\sin^2\theta \quad \text{or} \quad x=a\sin^3\theta$$

and

$$y^{2/3}=a^{2/3}\cos^2\theta \quad \text{or} \quad y=a\cos^3\theta$$

$\because$ the point is $(a\sin^3\theta,\ a\cos^3\theta)$.

Exercise XV

1. Find the equations of the tangent and the normal at *any point* to the following curves :

(i) $y^2=4ax$. (ii) $a^2y=x^3$.
(iii) $x^3+y^3=3axy$. (iv) $x=at^2$, $y=2at$.
(v) $x=a\sec\theta$, $p=b\tan\theta$.
(vi) $x=a(t+\sin t)$, $y=a(1-\cos t)$. *(K.U. Nov. 1976)*

2. Find the equations of tangents and normals to the following curves at the points indicated against each :

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x', y') and $(a \cos \theta, b \sin \theta)$.

(ii) $xy=c^2$ at (x', y').

(iii) $3y^2=x^3$ at $(3, 3)$.

(iv) $y=\frac{x^2}{4a}$ at $(2a, a)$.

(v) $x^2+y^2=2ax$ at $(2a, 0)$.

3. Show that the tangent to the curve $y=3x^4-4x^3+1$ is parallel to the x-axis at the point, where $x=1$.

4. Find the points on the curve

$$y=12x-x^3$$

at which tangents are parallel to the axis of x.

5. Show that $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y=be^{-x/a}$ at the point where it crosses the y-axis. *(K.U. Nov. 1976)*

6. Show that for all values of n

$$\frac{x}{a}+\frac{y}{b}=2$$

touches the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at the point (a, b).

(K.U. Old Course, 1976)

7. Find the equation of the tangent to the parabola $y^2=4x+5$ parallel to the line $2x-y=3$.

8. Find the equation of the normal to the curve $3x^2-y^2=14$ parallel to $x+3y=4$.

9. The tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ meets the axes of x and y at P and Q respectively. show that

$$OP+OQ=a.$$

10. Show that the portion of the tangent to the curve

$$x^{2/3}+y^{2/3}=a^{2/3}$$

which is intercepted between the axes is of constant length.

[**Hint.** Take (x, y) as the point of contact.] *(K.U., B.A., 1977)*

11. Find the coudition that the line

$$lx+my+n=0$$

may touch the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

12. Show that the condition thht the line $x \cos \alpha+y \sin \alpha=p$ may touch the curve $x^m y^n=a^{m+n}$ is

$$p^{m+n}\, m^m \,.\, n^n=(m+n)^{m+n} \,.\, a^{m+n} \,.\, \cos^m \alpha \,.\, \sin^n \alpha.$$

(K.U. B.A. (Pr.) 1977)

13. Find the condition that the line

$$x \cos \alpha + y \sin \alpha = p$$

may be a tangent to the curve

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1.$$

14. Show that any tanget to the hyperbola $xy = c^2$ forms a triangle of constant area with the axes.

15. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle θ with the x-axis, show that its equation is

$$y \cos \theta - x \sin \theta = a \cos 2\theta. \qquad (K.U.)$$

16. Prove that all points of the curve

$$y^2 = 4a \left(x + a \sin \frac{x}{a} \right)$$

at which the tangent is parallel to the x-axis lie on the parabola $y^2 = 4ax$.

[**Hint.** Put $\frac{dy}{dx} = 0$ and eliminate $\frac{x}{a}$]

17. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on $x^2y^2 = x^2 - y^2$.

18. Show that the tangents to the curve $x^3 + y^3 = 3axy$ at the points where it meets the parabola $y^2 = ax$ are parallel to the axis of y.

[**Hint.** Put $\frac{dx}{dy} = 0$ etc.]

7·6. Angle of Intersection of two curves

Def. *The angle of intersection of two curves is the angle between the tangents drawn to the curves at their point of intersection.*

Let us consider two curves $y = f(x)$ and $y = \phi(x)$ intersecting at a point $P(x_1, y_1)$. Then the angle between the respective tangents to the curves at the point P is called the *angle of intersection* of the two curves.

7·7 The following is the working rule for finding theangle between two curves

First step. Solve the equations of the two curves simultaneously in order to find the common point of intersection.

Second step. Find $\frac{dy}{dx}$ from the equations of the two curves separately.

Third step. Find the values of $\frac{dy}{dx}$ at the point of intersection found in the first step. Denote these values of $\frac{dy}{dx}$ by m_1 and m_2.

Fourth step. Use the formula

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$$

where θ is the angle between the two curves.

Note. The supplement of this angle is

$$\tan^{-1}\frac{m_2-m_1}{1+m_1m_2}.$$

Cor. 1. If the two curves cut *orthogonally* at $P(x_1, y_1)$ then
$$f'(x_1).\varphi'(x_1)=-1.$$

Cor. 2. If the two curves $y=f(x)$ and $y=\varphi(x)$ touch at $P(x_1, y_1)$ then $f'(x_1)=\varphi'(x_1)$.

7·8. To show that the curves
$$ax^2+by^2=1, \quad \text{and} \quad lx^2+my^2=1$$
cut at *right angles*, if
$$\frac{1}{a}-\frac{1}{b}=\frac{1}{l}-\frac{1}{m}.$$

Let $P(x_1, y_1)$ be the point of intersection of the given curves, then
$$ax_1^2+by_1^2=1 \quad \text{and} \quad lx_1^2+my_1^2=1.$$

Solving these equations for x_1^2 and y_1^2, we get
$$x_1^2=\frac{m-b}{am-bl}, \quad y_1^2=\frac{a-l}{am-bl}$$

From the equation of the first curve, we get
$$\frac{dy}{dx}=-\frac{ax}{by}.$$

Similarly, from the equation of the second curve
$$\frac{dy}{dx}=-\frac{lx}{my}.$$

Therefore, the slopes of the tangents to the two curves at $P(x_1, y_1)$ are respectively
$$-\frac{ax_1}{by_1} \text{ and } -\frac{lx_1}{my_1}.$$

The two tangents will be at right angles, if
$$\frac{alx_1^2}{bmy_1^2}=-1 \qquad (m_1m_2=-1)$$
i.e. if
$$alx_1^2=-bmy_1^2.$$

Now substituting the values of x_1^2 and y_1^2, we get
$$\frac{al(m-b)}{am-bl}=-\frac{bm(a-l)}{am-bl}$$
or
$$\frac{1}{a}-\frac{1}{b}=\frac{1}{l}-\frac{1}{m}.$$

Solved Examples

Example 1. Find the angle of intersection of the parabola $y^2=2x$ and the circle $x^2+y^2=8$.

Sol. Solving the two equations simultaneously, we get (2, 2) and (2, -2) as the two points of intersection.

Now from $\quad y^2=2x, \ \dfrac{dy}{dx}=\dfrac{1}{y}$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(2,\ 2)}=\frac{1}{2}=m_1 \text{ (say)}$$

Also from $\quad x^2+y^2=8, \ \dfrac{dy}{dx}=-\dfrac{x}{y}$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(2,\ 2)}=-1=m_2 \text{ (say)}$$

Hence
$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{\frac{1}{2}-(-1)}{1+\frac{1}{2}(-1)}=3$$

or
$$\theta=\tan^{-1}3.$$

Again, from equation (1), $\dfrac{dy}{dx}$ at $(2,\ -2)=-\frac{1}{2}=m_1$

and ,, ,, (2), $\dfrac{dy}{dx}$ at $(2,\ -2)=1=m_2$

$$\therefore \quad \tan\theta'=\frac{m_1-m_2}{1+m_1m_2}=\frac{-\frac{1}{2}-1}{1-\frac{1}{2}}=-3$$

or
$$\theta'=\tan^{-1}(-3).$$

Example 2. Show that the two curves $x^3-3xy^2=2a^3$ and $y^3-3x^2y=11a^3$ cut orthogonally at $(2a, -a)$. *(K.U. 1958)*

Sol. The two equations of the curves are

$$x^3-3xy^2=2a^3 \qquad \ldots(1)$$

and
$$y^3-3x^2y=11a^3 \qquad \ldots(2)$$

From (1), $\quad \dfrac{dy}{dx}=-\dfrac{x^2-y^2}{2xy}$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(2a,\ -a)}=-\frac{3}{4}=m_1 \text{ (say)}$$

Also from (2), $\quad \dfrac{dy}{dx}=\dfrac{2xy}{y^2-x^2}$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(2a,\ -a)}=\frac{4}{3}=m_2 \text{ (say)}$$

Now $\quad m_1, m_2=-\frac{3}{4}\times\frac{4}{3}=-1.$

Hence the two curves intersect at right angles at $(2a, -a)$.

Exercise XVI

Find the angle of intersection of the following curves :

1. $x^2-y^2=a^2$, $x^2+y^2=2a^2$.
2. $x^2-y^2=a^2$, $x^2+y^2=\overline{2a^2}$.
3. $x^2+y^2=2a^2$, $y^2=ax$.
4. $y^2=4ax$, $x^2=4ay$.
5. $y^2=ax$, $x^2=by$.

$x^2 = ay$

6. $y^2=ax$, $x^3+y^3=3axy$.
7. Show that the curves

$$x^3-3xy^2+2=0 \quad \text{and} \quad 3x^2y-y^3=2$$

cut orthogonally.

8. The curves $xy=c_1$ and $x^2y=c_2$ pass through the point (3, 4). Find the angle at which they intersect.

9. Show that the parabolas $x^2=ay$ and $y^2=2ax$ intersect on the curve $x^3+y^3=3axy$. Also find the angles between each ~~point~~ paw at the points of intersection.

10. Show that the curves

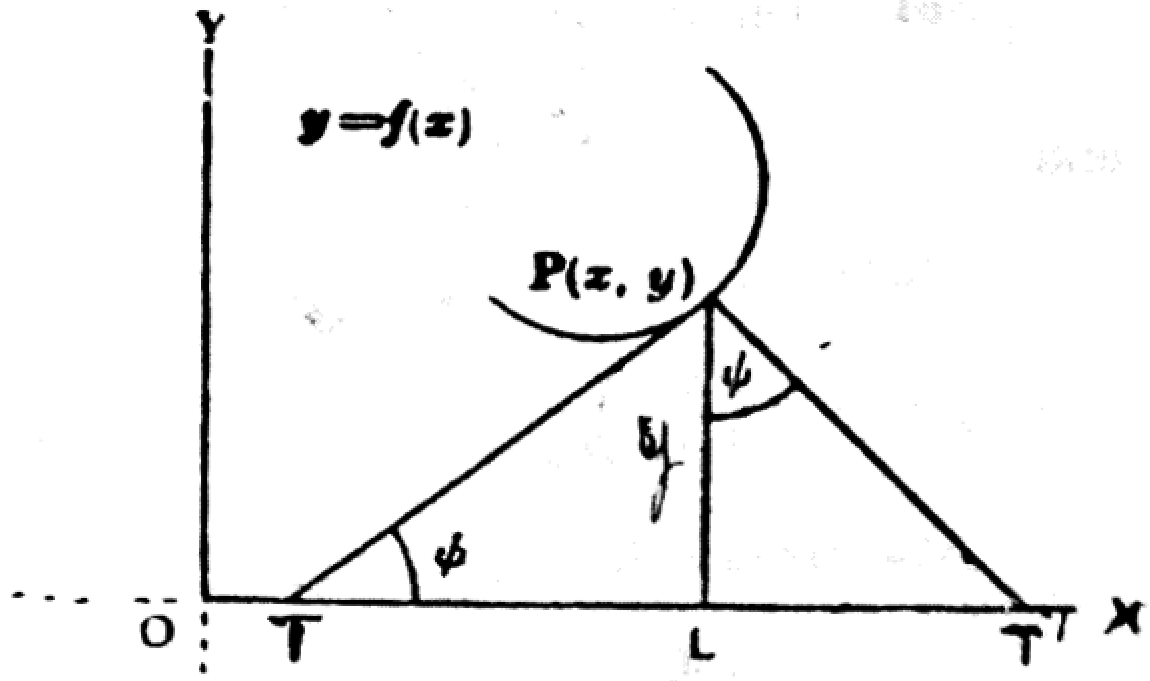
$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \text{ and } \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$$

intersect at right angles.

[**Hint.** Proceed as in 7·8.]

7·9. Lengths of Tangent, Normal, Subtangent and Subnormal

Let PT be the tangent to the curve $y=f(x)$ at any point P(x, y) making angle ψ with the positive side of the x-axis and let PT′ be the normal to the curve meeting the x-axis at T′. Draw PT ⊥ upon x-axis, so that PL$=y$. Then we have the following definitions :



1. Pt, the portion of the tangent intercepted between the point of contact and the axis of x, is defined as the *length of tangent*.

2. PT′, the portion of the normal intercepted between the point of contact and the axis of x, is defined as the ***length of the normal***.

3. TL, the projection of PT on x-axis, is defined as the length of the *subtangent*.

4. T′L, the projection of PT′ on x-axis, is defined as the length of the *sub-normal*.

Now $\angle PTL=\psi$

$\therefore$ $\tan\psi=\dfrac{dy}{dx}=y_1$

Also $\angle LPT'=90°-\angle LTP=90°-(90°-\psi)=\psi$

and $PL=y$.

Therefore :

1. Length of the tangent $=PT=y\operatorname{cosec}\psi$

$$y=\sqrt{1+\cot^2\psi}$$
$$=y\sqrt{1+\frac{1}{y_1^2}}$$
$$=\frac{y}{y_1}\sqrt{1+y_1^2}.$$

2. Length of the normal $=PT'=y\sec\psi$

$$=y\sqrt{1+\tan^2\psi}=y\sqrt{1+y_1^2}.$$

3. Length of the subtangent

$$=TL=y\cot\psi$$
$$=\frac{y}{y_1}.$$

4. Length of the subnormal

$$=T'L=y\tan\psi=yy_1.$$

Solved Examples

Example 1. Show that in any Cartesian curve

$$\left[\frac{\text{Tangent}}{\text{Normal}}\right]^2=\frac{\text{Subtangent}}{\text{Subnormal}}.$$ *(K.U. Inter., 1957)*

Sol. Tangent $=\dfrac{y}{y_1}\sqrt{1+y_1^2}$

Normal $=\sqrt{1+y_1^2}$

Subtangent $=\dfrac{y}{y_1}$

and Subnormal $=yy_1$.

Now L.H.S. $=\left[\dfrac{\text{Tangent}}{\text{Normal}}\right]^2$

$$=\left[\frac{\frac{y}{y_1}\sqrt{1+y_1^2}}{y\sqrt{1+y_1^2}}\right]^2=\frac{1}{y_1^2}$$

$$\text{R.H.S.} = \frac{\text{Subtangent}}{\text{Subnormal}} = \frac{\frac{y}{y_1}}{yy_1} = \frac{1}{y_1^2}.$$

Hence L.H.S. = R.H.S.

Example 2. Show that for the curve $y = be^{x/c}$ the subtangent is of constant length, and the subnormal varies as the square of the ordinate. *(P.U.)*

Sol. Here $\qquad y = be^{x/c}.$

$$\therefore \qquad \frac{dy}{dx} = \frac{b}{c}\, e^{x/c} = \frac{1}{c}\, y \qquad (\because\ y = be^{x/c})$$

$$\therefore \qquad \text{Subtangent} = \frac{y}{\frac{dy}{dx}} = \frac{y}{\frac{y}{c}} = c \text{ (constant).}$$

$$\text{Again, subnormal} = y\frac{dy}{dx} = y.\frac{y}{c} = \frac{y^2}{c}.$$

$$\therefore \qquad \frac{\text{Subnormal}}{y^2} = \frac{1}{c} \text{ (constant).}$$

Hence the *subnormal varies as the square of the ordinate.*

Exercise XVII

1. Show that in any cartesian curve :

   (*i*) Subtangent × subnormal = (ordinate)^2.

   (*ii*) (Normal)^2 = (subnormal)^2 + (ordinate)^2.

2. Show that in the curve $y = be^{-\frac{x}{a}}$ the subnormal varies as the square of ordinate, and that in the curve $y = a^x$, the subtangent is constant.

3. Show that the tangent to the curve

$$\frac{x + \sqrt{a^2 - y^2}}{a} = \log \frac{a + \sqrt{a^2 - y^2}}{y}$$

is constant and equal to a.

$1 = \frac{yy_1}{a + \sqrt{a^2 - y^2}} \times - \frac{ay_1}{y}$

$\frac{y_1}{y} [$

4. Show that in the curve

$$x - a + \sqrt{b^2 - y^2} = b \log \{b + \sqrt{b^2 - y^2}\}$$

the sum of the subnormal and subtangent is constant.

5. In the curve

$$x = a(\cos t + \log \tan \tfrac{1}{2}t),\ y = a \sin t$$

prove that the portion of the tangent intercepted between the x-axis and the curve is of constant length.

6. Find the lengths of tangent, normal, subtangent and subnormal at the point θ on the curve

$$x = a \cos^3 \theta,\ y = a \sin^3 \theta.$$

7. Prove that the sum of the tangent and subtangent at any point of the curve

$$e^{y/a}=x^2-a^2$$

varies as the product of the corresponding co-ordinates.

(*K.U. 1975*)

8. Show that the subtangent at any point of curve

$$x^m y^n=a^{m+n}$$

varies as the abscissa.

9. Show that the subnormal at any point of the curve

$$y^2x^2=a^2(x^2-a^2)$$

varies inversely as the cube of its abscissa.

10. Prove that for the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the length of the normal varies inversely as the perpendicular from the origin upon the tangent. (*P.U.*)

11. Prove that for the catenary $y=c \cosh \dfrac{x}{c}$ the perpendicular dropped from the foot of the ordinate upon the tangent is of constant length.

12. For the curve $y=a \cosh \left(\dfrac{x}{a}\right)$, prove that the length of the portion of the normal intercepted between the curve and the x-axis is $\dfrac{y^2}{a}$.

(*K U. B.A., 1974*)

$$\frac{1}{a}\left\{1-\frac{y y_1}{\sqrt{a^2-y^2}}\right\}=\frac{-y}{a+\sqrt{a^2-y^2}}\left[\frac{y_1}{\sqrt{a^2-y^2}}+\frac{1}{y^2}\left(a+\sqrt{a^2-y^2}\right)y_1\right]$$

8

Integration

8·1. Integration as the reverse of differentiation

Integration is the process by which we can find a function whose differential coefficient is given. It is thus a process which is the inverse of differentiation. For instance, let us take the case of x^n. If we differentiate x^n w.r.t. n we get nx^{n-1}. Now nx^{n-1} is called the differentiate co-efficient of x^n w.r.t. x, whereas x^n is the integral of nx^{n-1}

The process of *differentiating* x^n with respect to x is denoted by the equation

$$\frac{d}{dx}(x^n)=nx^{n-1}$$

while the one of *integrating* nx^{n-1} with respect to x is denoted by the equation

$$\int nx^{n-1}\,dx=x^n.$$

Thus, the symbol $\int dx$ denotes the process of *integration with respect to* x just as the symbol $\frac{d}{dx}$ denote the process of *differentiation with respect to* x.

8·2. In the preceeding Article we have defined *integration* as the inverse process of differentiation. Consequently, if we are required to integrate cos x w.r.t. x, we are evidently required to find out a function whose differential co-oefficint w.r.t. x is cos x. Obviously, this function is sin x. This can be *symbolically* represented as

$$\int \cos x\,dx=\sin x$$

and is read as ;

"integral of cos x w.r.t. x=sin x".

8·3. We give below some of the *elementary* fundamental formulae of integration and the student is advised to commit these to memory. Most of these have been proved in the forthcoming articles.

I. Algebraic Functions

1. $\int x^n\,dx=\frac{x^{n+1}}{n+1}$. (provided $n\neq-1$)

2. $\int (ax+b)^n \, dx = \dfrac{(ax+b)^{n+1}}{a(n+1)}$, (provided $n \neq -1$)

3. $\int \dfrac{dx}{x} = \log x$.

4. $\int \dfrac{dx}{ax+b} = \dfrac{1}{a} \log (ax+b)$.

5. $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a} \tan^{-1} \dfrac{x}{a}$.

6. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \dfrac{x}{2}$.

7. $\int \dfrac{dx}{x\sqrt{x^2-a^2}} = \dfrac{1}{a} \sec^{-1} \dfrac{x}{a}$.

II. Trigonometric Functions

1. $\int \sin x \, dx = -\cos x$.
2. $\int \cos x \, dx = \sin x$.
3. $\int \sec^2 x \, dx = \tan x$.
4. $\int \text{cosec}^2 x \, dx = -\cot x$.
5. $\int \sec x \tan x \, dx = \sec x$.
6. $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x$.
7. $\int \tan x \, dx = \log \sec x$.
8. $\int \cot x \, dx = \log \sin x$.

III. Exponential Functions

1. $\int e^{mx} \, dx = \dfrac{e^{mx}}{m}$.
2. $\int a^{mx} \, dx = \dfrac{a^{mx}}{m} \log a$.

8·4. Two important formulae

1. $\int [f(x)]^n f'(x) \, dx = \dfrac{[f(x)]^{n+1}}{n+1}$.

2. $\int \dfrac{f'(x)}{f(x)} \, dx = \log f(x)$.

Proof :

1. $\dfrac{d}{dx}\left[\dfrac{\{f(x)\}^{n+1}}{n+1}\right] = \dfrac{[f(x)]^n f'(x)(n+1)}{n+1} = [f(x)^n f'(x)]$

Hence $\int [f(x)]^n f'(x) \, dx = \dfrac{[f(x)]^{n+1}}{n+1}$.

2. $\dfrac{d}{dx}\left[\log f(x)\right] = \dfrac{1}{f(x)} \times f'(x) = \dfrac{f'(x)}{f(x)}$

Hence $\int \dfrac{f'(x)}{f(x)} \, dx = \log f(x)$.

8·5. General Rules for Integration

Theorem 1. *A constant factor can be taken out of the integral sign without affecting the value of the integral.*

Or, symbolically,

$$\int af(x) \, dx = a \int f(x) \, dx \quad \dots(1)$$

Proof. Differentiating both sides of (1) w.r.t. x, we have

$$\frac{d}{dx}\left[\int af(x)\,dx\right]=\frac{d}{dx}\left[a\int f(x)\,dx\right]$$

Now $\frac{d}{dx}\left[\int af(x)\,dx\right]=af(x)$

$$\left[\because \quad \frac{d}{dx} \text{ and } \int \text{ cancel each other}\right]$$

and $\frac{d}{dx}\left[a\int f(x)\,dx\right]=a\frac{d}{dx}\left[\int f(x)\,dx\right]=af(x)$

Hence $\int af(x)\,dx=a\int f(x)\,dx.$

Example. $\int 6x^7\,dx=6\int x^7\,dx=\frac{6}{8}\,x^8=\frac{3}{4}\,x^8.$

Theorem II. *The integral of the algebraic sum of two functions is equal to the algebraic sum of their integrals.*

Or, symbolically,

$$\int [f(x)+\varphi(x)]\,dx=\int f(x)\,dx+\int\varphi(x)\,dx \qquad \ldots(1)$$

Proof. Differentiating both sides of (1) w.r.t. x, we have

$$\frac{d}{dx}\left[\int\left\{f(x)+\varphi(x)\right\}dx\right]=\frac{d}{dx}\left[\int f(x)\,dx+\int\varphi(x\,\,dx\right]$$

or $$f(x)+\varphi(x)=\frac{d}{dx}\int f(x)\,dx+\frac{d}{dx}\int\varphi(x)\,dx$$

$$=f(x)+\varphi(x).$$

Hence the theorem.

Note. This theorem can be extended to any number of functions. Thus :

$$\int\{f_1(x)\pm f_2(x)\pm f_3(x)+\ldots\ldots\}\,dx$$
$$=\int f_1(x)\,dx\pm\int f_2(x)\,dx\pm\int f_3(x)\,dx\pm\ldots\ldots$$

Example 1. $\int (x^3+6x^2)\,dx=\int x^3\,dx+6\int x^2\,dx$

$$=\frac{x^4}{4}+\frac{6x^3}{3}=\frac{x^4}{4}+2x^3.$$

2. $\int (3x^2+7x+5)\,dx=\int 3x^2\,dx+\int 7x\,dx+\int 5dx$

$$=3\int x^2\,dx+7\int x\,dx+5\int dx$$

$$=\frac{3x^2}{3}+\frac{7x^2}{2}+5x$$

$$=x^3-\frac{7}{2}\,x^2+5x.$$

8 6. Proofs of some fundamental formulae

We give below the proofs of some of the most important formulae of integration, and the student is advised to commit these

to memory as he will be required to apply them in solving questions on integration very frequently.

I. $$\int (ax+b)^n\, dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} \qquad (n \neq -1)$$

The proof of this formulae (which can be got by differentiating the R.H.S.) is left to the student as an exercise.

Cor. $$\int x^n\, dx = \frac{x^{n+1}}{n+1}.$$

This is a special case of formulae I and can be proved by differentiating the R.H.S.

II. $$\int \frac{dx}{ax+b} = \frac{1}{a} \log (ax+b).$$

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Cor. $$\int \frac{dx}{d} = \log x.$$

[These can be proved by the same way as formula (I).]

Illustrations

(Illustrations on formula I)

(*i*) $$\int (3x+6)^3\, dx = \tfrac{1}{3}\, \frac{(3x+6)^4}{4} = \frac{1}{12}\, (3x+4)^4.$$

(*ii*) $$\int \frac{dx}{\sqrt{2x+5}} = \int (2x+5)^{-1/2}\, dx$$

$$= \tfrac{1}{2}\, \frac{(2x+5)^{-1/2+1}}{-\frac{1}{2}+1} = \tfrac{1}{2} \cdot \frac{(2x+5)^{1/2}}{\frac{1}{2}}$$

$$= \sqrt{2x+5}.$$

(*iii*) $$\int \frac{dx}{(8-9x)^3} = \int (8-9x)^{-3}\, dx$$

$$= -\frac{1}{9}\, \frac{(8-9x)^{-3+1}}{-3+1} = \frac{1}{18} \cdot \frac{1}{(8-9x)^2}.$$

(*iv*) $$\int \frac{dx}{x^4} = \int x^{-4}\, dx = \frac{x^{-4+1}}{-4+1} = -\frac{1}{3} \cdot \frac{1}{x^3}.$$

(Illustrations on formula II)

(*i*) $$\int \frac{dx}{3x+4} = \frac{1}{3} \log (3x+4).$$

(*ii*) $$\int \frac{dx}{1-2x} = -\frac{1}{2} \log (1-2x) = \frac{1}{2} \log \frac{1}{1-2x}.$$

$$\left(\text{Please note that } -\log x = \log \frac{1}{x} \right)$$

III. (*a*) $\int \sin^2 x\, dx = \frac{1}{2}(x - \frac{1}{2} \sin 2x)$.

Proof. $\sin^3 x = \dfrac{1-\cos 2x}{2}$.

$$\therefore \quad \int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx$$

$$= \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x\, dx = \frac{1}{2}\, x - \frac{1}{2}\, \frac{\sin 2x}{2}$$

$$= \tfrac{1}{2}(x - \tfrac{1}{2} \sin 2x).$$

III. (*b*) $\int \cos^2 x\, dx = \frac{1}{2}(x + \frac{1}{2} \sin 2x)$.

The proof of this formula is left as an exercise for the student.

IV. (*a*) $\int \tan^2 x\, dx = \tan x - x$.

Proof. $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$

$$= \int \sec^2 x\, dx - \int dx = \tan x - x.$$

IV. (*b*) $\int \cot^2 x\, dx = -\cot x - x$.

The proof is left as an exercise for the student.

V. (*a*) $\int \sin^2 x\, dx = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x$.

Proof. We have $\sin 3x = 3 \sin x - 4 \sin^3 x$

$$\therefore \quad \sin^3 x = \tfrac{3}{4} \sin x - \tfrac{1}{4} \sin 3x$$

Hence $\int \sin^3 x\, dx = \int (\frac{3}{4} \sin x - \frac{1}{4} \sin 3x)\, dx$

$$= \tfrac{3}{4} \int \sin x\, dx - \tfrac{1}{4} \int \sin 3x$$

$$= -\frac{3}{4} \cos x - \frac{1}{4}\left(-\frac{\cos 3x}{3}\right)$$

$$= \frac{1}{12} \cos 3x - \frac{3}{4} \cos x.$$

V. (*b*) $\int \cos^3 x\, dx = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x$.

The proof is left as an exercise for the student.

Note. *Other fundamental formulae will be proved at their proper places.*

8·7. Constant of Integration

We know that

$$\frac{d}{dx}(x^2) = 2x \qquad \therefore \int 2x\, dx = x^2$$

Also $$\frac{d}{dx}(x^2+3) = 2x \quad \therefore \int 2x\, dx = x^2 + 3$$

and $$\frac{d}{dx}(x^2+c) = 2x \quad \therefore \int 2x\, dx = x^2 + c.$$

This shows that x^2, x^2+3 and x^2+c are the integrals of the same function $2x$. Hence it is necessary to add an arbitrary constant c to the result of integration.

Therefore, if $\dfrac{d}{dx} f(x)=\varphi(x)$

then $\int \varphi(x)dx=f(x)+c.$

This constant "c" is called the constant of integration. Though it is frequently omitted from the result, its presence is always understood.

8·8. Definite Integrals

Definition. If $\varphi(x)$ is a function such that

$$\frac{d}{dx}\varphi(x)=f(x)$$

then the *definite integral* of $f(x)$ from a to $b=\varphi(b)-f(a)$.

This is denoted by the symbol

$$\int_a^b f(x)\,dx$$

so that $$\int_a^b f(x)\,dx=\Big[\varphi(x)\Big]_a^b=\varphi(b)-\varphi(a)$$

where $$\frac{d}{dx}\varphi(x)=f(x).$$

"b" is called the *upper limit* and "a" the *lower limit* of the definite integral.

8·9. Rule for finding the definite integral

1. Find the *indefinite* integral.
2. Substitute for x the upper limit and then the lower limit.
3. Take the difference between the two resulting expressions in (2).

Illustrations

(*i*) $$\int_a^b x^3\,dx=\left[\frac{x^4}{4}\right]_a^b=\frac{b^4}{4}-\frac{a^4}{4}.$$

(*ii*) $$\int_0^{\frac{\pi}{2}}\cos x\,dx=\Big[\sin x\Big]_0^{\frac{\pi}{2}}$$

$$=\sin\frac{\pi}{4}-\sin 0=1.$$

Solved Examples

Example 1. $\displaystyle\int\frac{3x^2}{1-x}\,dx.$

Sol. $\dfrac{3x^2}{1-x}=-3x-3+\dfrac{3}{1-x}.$

(Please note this step)

$$\therefore \quad \int \frac{3x^2\,dx}{1-x} = \int \left(-3x-3+\frac{3}{1-x}\right)dx$$

$$= -3\int x\,dx - 3\int dx + 3\int \frac{dx}{1-x}$$

$$= -\tfrac{3}{2}x^2 - 3x - 3\log(1-x).$$

Example 2. $\int x\sqrt{a+x}\,dx.$

Sol. $$\int x\sqrt{a+x}\,dx = \int (a+x-a)\sqrt{a+x}\,dx$$

$$= \int (a+x)\sqrt{a+x}\,dx - \int a\sqrt{a+x}\,dx$$

$$= \int (a+x)^{3/2}\,dx - a\int (x+a)^{1/2}\,dx$$

$$= \frac{(a+x)^{5/2}}{\frac{5}{2}} - a\,\frac{(x-a)^{3/2}}{\frac{3}{2}}$$

$$= \frac{2}{5}(a+x)^{5/2} - \frac{2a}{3}(x+a)^{3/2}.$$

Example 3. $\int \frac{x\,dx}{\sqrt{a+x}}.$

Sol. $$\int \frac{x\,dx}{\sqrt{a+x}} = \int \frac{(a+x-a)dx}{\sqrt{a+x}}$$

$$= \int \frac{a+x}{\sqrt{a+x}}\,dx - \int \frac{a\,dx}{\sqrt{a+x}}$$

$$= \int \sqrt{a+x}\,dx - a\int \frac{dx}{\sqrt{a+x}}$$

$$= \int (a+x)^{1/2}\,dx - a\int (a+x)^{-1/2}\,dx$$

$$= \tfrac{2}{3}(a+x)^{3/2} - 2a(a+x)^{1/2}.$$

Example 4. $\int \frac{dx}{1+\sin x}$ *(K.U. 1977)*

Sol. We have

$$\int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)\,dx}{1-\sin^2 x}$$

$$= \int \frac{(1-\sin x)\,dx}{\cos^2 x}$$

$$= \int \frac{dx}{\cos^2 x} - \int \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int \sec^2 x\,dx - \int \sec x \tan x\,dx$$

$$= \tan x - \sec x.$$

Example 5. $\int \sqrt{1+\sin x}\, dx$

Sol. $\int \sqrt{1+\sin x}\, dx$

$$=\int \sqrt{\cos^2 \frac{x}{2}+\sin^2 \frac{x}{2}+2 \sin \frac{x}{2} \cos \frac{x}{2}}\, dx$$

$$=\int \sqrt{\left(\cos \frac{x}{2}+\text{stn} \frac{x}{2}\right)^2}\, dx$$

$$=\int\left(\cos \frac{x}{2}+\sin \frac{x}{2}\right) dx$$

$$=\frac{\sin \frac{x}{2}}{\frac{1}{2}}-\frac{\cos \frac{x}{2}}{\frac{1}{2}}$$

$$=2\left(\sin \frac{x}{2}-\cos \frac{x}{2}\right).$$

Example 6. $\int \sin 3x \sin x\, dx.$

Sol. $\sin 3x \sin x=\frac{1}{2}(2 \sin 3x \sin x)$

$$=\tfrac{1}{2}(\cos 2x-\cos 4x)$$

$\therefore$ $\int \sin 3x \sin x\, dx=\frac{1}{2}\int(\cos 2x-\cos 4x)\, dx$

$$=\tfrac{1}{2}\int \cos 2x\, dx-\tfrac{1}{2}\int \cos 4x\, dx$$

$$=\frac{1}{2}\cdot\frac{\sin 2x}{2}-\frac{1}{2}\cdot\frac{\sin 4x}{4}$$

$$=\tfrac{1}{4}\sin 2x-\tfrac{1}{8}\sin 4x.$$

Exercise XVIII

1. Integrate the following w.r.t. x:
   x^7 ; $3x^2$; $x^{-1/2}$; $(x+8)^5$; $(2x+3)^5$.

2. Evaluate the following :

(i) $\int \frac{dx}{(3x+4)^5}$. (ii) $\int(x^3-x^4+2x^5)\, dx$.

(iii) $\int\left(3x+4x+\frac{5}{x}\right)dx$. (iv) $\int \frac{x^3+x^2+x-3}{x^5}\, dx$.

(v) $\int \frac{a+bx+cx^2}{cx^3}\, dx$. (vi) $\int \frac{dx}{x-5}$.

(vii) $\int \frac{dx}{a-bx}$. (viii) $\int(1-\theta+\theta^2)^2\, d\theta$.

(ix) $\int\left(x+\frac{1}{x}\right)^3 dx$. (x) $\int\left(\sqrt{x-1}+\frac{1}{\sqrt{x+1}}\right)dx$.

(xi) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$. (xii) $\int \frac{dx}{(ax+b)^4}$. (K U. 1964)

3. Evaluate :

(i) $\int \frac{x+5}{x-4}\,dx$. (ii) $\int \frac{4x+5}{x-3}\,dx$.

(iii) $\int \frac{3x\,dx}{3x+4}$. (iv) $\int \frac{x^3-7}{x+3}\,dx$.

(v) $\int \frac{2x^2+1}{2x+1}\,dx$. (vi) $\int \frac{x^2+2x+3}{x+1}\,dx$.

(vii) $\int \frac{x^3-27}{x-3}\,dx$. (viii) $\int \frac{x\,dx}{\sqrt{x+3}}$.

(ix) $\int \frac{x+b}{\sqrt{x+a}}\,dx$. (x) $\int x\sqrt{x+3}\,dx$.

4. Evaluate :

(i) $\int \cos\frac{x}{3}\,dx$. (ii) $\int \sin(\pi+x)\,dx$.

(iii) $\int \cos\left(\frac{\pi}{2}+x\right)dx$. (iv) $\int \sec^2 x\,dx$.

(v) $\int \sec^2 4x\,dx$. (vi) $\int \frac{dx}{\cos^2 ax}$.

(vii) $\int \tan^2 x\,dx$. (viii) $\int \cos^3 x\,dx$.

(ix) $\int (\sin x-\cos x)^2\,dx$. (x) $\int \frac{dx}{\sin^2 x\cos^2 x}$.

$\left[\textbf{Hint.}\ \frac{1}{\sin^2 x\cos^2 x}=\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}\right]$

(xi) $\int \frac{d\theta}{1+\cos 2\theta}$. (xii) $\int (\tan^2 x-\cot^2 x)\,dx$.

(xiii) $\int \sin 3x\cos 2x\,dx$. (xiv) $\int \cos 2x\cos 3x\,dx$.

(*K.U. 1964*)

(xv) $\int \cos px\sin qx\,dx$. $(p>q)$.

(xvi) $\int \frac{dx}{1+\cos x}$. (xvii) $\int \frac{dx}{1-\cos x}$.

(xviii) $\int \frac{\sin x}{1+\sin x}\,dx$. (xix) $\int \sqrt{1+\sin x}\,dx$.

(*K.U. 1976*)

(xx) $\int \sqrt{1+\cos x}\,dx$. $\left[\textbf{Hint.}\ 1+\cos x=2\cos^2\frac{x}{2}\right]$

5. Evaluate :

(i) $\int e^{-x}\,dx$. (ii) $\int e^{-ax}\,dx$. (iii) $\int 10^x\,dx$.

(iv) $\int \frac{e^{2x}+1}{e^x} dx.$ (v) $\int (e^x+e^{-x})^2 dx.$

(vi) $\int (x^n+n^x-1) dx.$

5. Find the value of :

(i) $\int_0^{\pi/4} \sin x \cos x \, dx.$ (ii) $\int_0^{\pi/10} \sin 5x \, dx.$

(iii) $\int_0^{\pi/4} \tan^2 x \, dx.$ (iv) $\int_0^1 \frac{3x^2+1}{2x+1} dx.$

(v) $\int_0^7 \frac{dx}{4x+7}.$ (vi) $\int_0^{\pi} \sin^2 x \, dx.$

(vii) $\int_0^{\pi/3} \sec^2 x \, dx.$ (viii) $\int_{-\pi/6}^{\pi/6} \operatorname{cosec}^2 \theta \, d\theta.$

8·10. Integration by Substitution

So far we have dealt with a number of simple functions which we have been able of integrate from our previous knowledge of Differential Calculus. For instance, we know that the integration of $\cos x$ w.r.t. x is $\sin x$ because the differential co-efficient of $\sin x$ w.r.t. x is $\cos x$. There are, however, a number of other functions which cannot be readily integrated because we do not know functions of which these are the derivatives. Thus, there are several *devices* which are employed in order to integrate such functions *conveniently*. One of these devices is called "*the method of substitution*", or "*integration by changing the variable*".

Before we apply this method we give below a new meaning to the symbol dx.

If $y=f(x)$, we have

$$\frac{dy}{dx}=f'(x).$$

This can be written as

$$dy=f'(x) \, dx$$

where dx is called the *differential* of x and $f'(x)$ is called the *co-efficient* of the differential. For this very reason, $f'(x)$ is called the *differential co-efficient*. In changing the variable of integration, we have to replace the differential dx by the *new differential of new variable*. The method is explained below by means of an example,

$$\int \frac{\log x}{x} dx$$

Putting $\log x=z$, we have

$$\frac{1}{x} dx=dz$$

$$\therefore \quad \int \frac{\log x}{x} dx=\int z dz=\frac{z^2}{2}=\frac{(\log x)^2}{2}.$$

Here note that we have changed not only the differential dx but also $\dfrac{dx}{x}$ by one and the same substitution $\log x = z$.

8·11. Integration of some Fundamental Functions

We integrate below some of the fundamental functions by the *method of substitution*. The student is advised to learn the proofs of these *by heart*, as he cannot dispense with them in solving important questions on integrations.

I. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$

Sol. Put $x = a\tan\theta$, then $dx = a\sec^2\theta\, d\theta$

$$\therefore \quad \int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2(1+\tan^2\theta)} = \frac{1}{a}\int \frac{\sec^2\theta\, \theta\theta}{\sec^2\theta}$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\,\theta = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

II. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}.$

Sol. Putting $x = a\sin\theta$, we have

$$dx = a\cos\theta\, d\theta$$

$$\therefore \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$

$$= \int \frac{a\cos\theta\, d\theta}{a\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta\, d\theta}{\cos\theta} = \int d\theta$$

$$= \theta = \sin^{-1}\frac{x}{a}.$$

III. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}.$

Sol. Putting $x = a\cos\theta$, so that

$$dx = a\sec\theta\tan\theta\, d\theta$$

$$\therefore \quad \int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{a\sec\theta\,\sqrt{a^2\sec^2\theta-a^2}}$$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{a^2\sec\theta\tan\theta} = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\,\theta = \frac{1}{a}\sec^{-1}\frac{x}{a}.$$

IV. $\displaystyle\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

Sol. Putting $x = a\sin\theta$, we get

$$dx = a\cos\theta\, d\theta$$

$$\therefore \int \sqrt{a^2-x^2}\ dx=\int \sqrt{a^2-a^2 \sin^2 \theta}\ .\ a \cos \theta\ d\theta$$

$$=\int a^2 \sqrt{a-\sin 2\theta}\ .\ \cos \theta\ d\theta=a^2\int \cos^2 \theta\ d\theta$$

$$=a^2\int \frac{1+\cos^2 \theta}{2}\ d\theta=\frac{a^2}{2}\int(1+\cos 2\theta)\ d\theta$$

$$=\frac{a^2}{2}\int d\theta=\frac{a^2}{2}\int \cos 2\theta\ d\theta$$

$$=\frac{a^2}{2}\theta+\frac{a^2}{2}\cdot\frac{\sin 2\theta}{2}$$

$$=\frac{a^2}{2}\theta+\frac{a^2}{2}\sin\theta\cos\theta$$

$$=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{a^2}{2}\cdot\frac{x}{a}\ \frac{\sqrt{a^2-x^2}}{a}$$

$$\left[\because\ \cos\theta=\sqrt{1-\sin^2\theta}=\frac{\sqrt{a^2-x^2}}{a}\right]$$

$$=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x\sqrt{a^2-x^2}}{2}.$$

V. $\int \sqrt{\frac{a+x}{a-x}}\ dx=-a\cos^{-1}\frac{x}{a}-\sqrt{a^2-x^2}.$

Sol. Putting $x=a\cos 2\theta$, we get

$$dx=-2a\sin 2\theta\ d\theta$$

$$\therefore \int\sqrt{\frac{a+x}{a-x}}\ dx=\int\sqrt{\frac{a(1+\cos 2\theta)}{a(1-\cos 2\theta)}}\times -2a\sin 2\theta\ d\theta$$

$$=-2a\int\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\ (\sin 2\theta)\ d\theta$$

$$=-2a\int\sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}\times 2\sin\theta\cos\theta\ d\theta$$

$$=-4a\int\frac{\cos\theta}{\sin\theta}\sin\theta\cos\theta\ d\theta$$

$$=-4a\int\cos^2\theta\ d\theta=-4a\int\frac{1+\text{cas}\ 2\theta}{2}\ d\theta$$

$$=-2a\int d\theta-2a\int\cos 2\theta\ d\theta$$

$$=-2a\theta-2a\frac{\sin 2\theta}{2}=-2a\theta-a\sin 2\theta$$

$$=-2a\cdot\frac{1}{2}\cos^{-1}\frac{x}{a}-a\frac{\sqrt{a^2-x^2}}{2}$$

$$\left(\because\ \sin 2\theta=\frac{\sqrt{a^2-x^2}}{2}\right)$$

$$=-a\cos^{-1}\frac{x}{a}-\sqrt{a^2-x^2}.$$

VI. $\int \sec x\,dx=\log(\sec x+\tan x)$

$$=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

Sol. $\displaystyle\int \sec x\,dx=\int\frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\,dx$

$$=\int\frac{\sec^2 x+\sec x\tan x}{\tan x+\sec x}\,dx$$

Putting $\tan x+\sec x=z$, we get

$$(\sec^2 x+\sec x\tan x)\,dx=dz$$

$$\therefore\ \int\frac{\sec^2 x+\sec x\tan x}{\tan x+\sec x}\,dx=\int\frac{dz}{z}=\log z$$

$$=\log(\sec x+\tan x)\ldots\ldots\ldots$$

Second Form

$$\log(\sec x+\tan x)=\log\left[\frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}+\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}\right]$$

$$\left(\because\ \cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\ \text{and}\ \tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}\right)$$

$$=\log\frac{\left(1+\tan\frac{x}{2}\right)^2}{1-\tan^2\frac{x}{2}}$$

$$=\log\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$$

$$=\log\frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\ \tan\frac{x}{2}}\quad\left(\because \tan\frac{\pi}{4}=1\right)$$

$$=\log\tan\left(\frac{\pi}{2}+\frac{x}{2}\right).$$

VII. $\int \operatorname{cosec} x\, dx=\log\tan\frac{x}{2}.$

Sol. $$\int \operatorname{cosec} x\, dx=\int\frac{dx}{\sin x}$$

$$=\int\frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\int\frac{\sec^2\frac{x}{2}\, dx}{2\sin\frac{x}{2}\cos\frac{x}{2}\sec^2\frac{x}{2}}$$

$$=\int\frac{\frac{1}{2}\sec^2\frac{x}{2}\, dx}{\sin\frac{x}{2}\sec\frac{x}{2}}$$

$$=\int\frac{\frac{1}{2}\sec^2\frac{x}{2}\, dx}{\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\int\frac{\frac{1}{2}\sec^2\frac{x}{2}\, dx}{\tan\frac{x}{2}}$$

Putting $\tan\frac{x}{2}=u$, we have

$$\frac{1}{2}\sec^2\frac{x}{2}dx=du$$

$$\therefore\quad\int\frac{\frac{1}{2}\sec^2\frac{x}{2}\, dx}{\tan\frac{x}{2}}=\int\frac{du}{u}=\log u=\log\tan\frac{x}{2}.$$

Second Form :

$$\int \operatorname{cosec} x \, dx = -\int \frac{-\operatorname{cosec} x (\operatorname{cosec} x + \cot x)}{\operatorname{cosec} x + \cot x} dx$$

$$= -\int \frac{-\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x + \cot x} dx$$

$$= -\log (\operatorname{cosec} x + \cot x).$$

VIII. $\int \frac{f'(x)}{f(x)} dx = \log f(x).$

Sol. Let $f(x) = z$, so that

$$f'(x) \, dx = dz$$

$$\therefore \quad \int \frac{f'(x) \, dx}{f(x)} = \int \frac{dz}{z} = \log z = \log f(x).$$

Note. This is very important result and may be expressed in words as under :

"The integral of a function whose numerator is the differential co-efficient of the denominor is the logarithm of the denominator."

Solved Examples

Example 1. $\int \tan x \, dx.$

Sol. $\int \tan x \, dx = \int \frac{\sin x \, dx}{\cos x}.$

Putting $\cos x = z$, we have

$$-\sin x \, dx = dz \quad \text{or} \quad \sin x \, dx = -dz$$

$$\therefore \quad \int \frac{\sin x}{\cos x} dx = -\int \frac{dz}{z} = -\log z = -\log \cos x$$

$$= \log \frac{1}{\cos x} = \log \sec x.$$

Example 2. $\int \frac{ax+b}{ax^2+2bx+c} dx.$

Sol. Putting $ax^2+2bx+c=z$, we get

$$2(ax+b)dx = dz$$

or

$$(ax+b)dx = \frac{dz}{2}$$

Hence $\int \frac{ax+b}{ax^2+2bx+c} dx = \int \frac{dz}{2/z} = \frac{1}{2} \int \frac{dz}{z}$

$$= \frac{1}{2} \log z = \frac{1}{2} \log (ax^2+2bx+c).$$

Example 3. $\dfrac{dx}{\sqrt{2ax-x^2}}$.

Sol. $\dfrac{dx}{\sqrt{2ax-x^2}}=\dfrac{dx}{\sqrt{a^2-(x^2-2ax+a^2)}}$

$$=\int\frac{dx}{\sqrt{a^2-(a-x)^2}}.$$

Putting $a-x=a\cos\theta$, we get
$dx=a\sin\theta\,d\theta$

$$\therefore\ \frac{dx}{\sqrt{2ax-x^2}}=\int\frac{a\sin\theta\,d\theta}{a\sqrt{1-\cos^2\theta}}=\int\frac{\sin\theta\,d\theta}{\sin\theta}=\int d\theta$$

$$=\theta=\cos^{-1}\left(\frac{a-x}{a}\right).$$

Note. Please note that substitutions for the integrals $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{\dfrac{a+x}{a-x}}$ are respectively $x=a\sin\theta$ (or $a\cos\theta$), $x=a\tan\theta$, $x=a\sec\theta$ and $x=a\cos 2\theta$.

Exercise XIX

Evaluate the following :

1. $\left(\text{Questions of the type }\int\dfrac{dx}{a^2+x^2}\right)$

(*i*) $\int\dfrac{dx}{x^2+4}$. (*ii*) $\int\dfrac{dy}{y^2+16}$

(*iii*) $\int\dfrac{dx}{4x^2+1}$. (*iv*) $\dfrac{dx}{16a^2+x^2}$.

(*v*) $\int\dfrac{dx}{x^2+4x+5}$. (*vi*) $\int\dfrac{dx}{x^2+8}$.

2. $\left(\text{Questions of the type }\int\dfrac{dx}{\sqrt{a^2-x^2}}\right)$

(*i*) $\int\dfrac{dx}{\sqrt{1-x^2}}$. (*ii*) $\int\dfrac{dx}{\sqrt{25-x^2}}$.

(*iii*) $\int\dfrac{dx}{\sqrt{2a^2-\frac{1}{2}x^2}}$. (Put $x=2a\sin\theta$)

(*iv*) $\int\dfrac{dx}{\sqrt{1-a^2x^2}}$. (*v*) $\int\dfrac{dx}{\sqrt{3-x^2}}$.

(*vi*) $\dfrac{dx}{\sqrt{2x-x^2}}$. (*vii*) $\int\dfrac{dx}{\sqrt{2-4x-x^2}}$.

3. (Questions of the type $\int \frac{dx}{x\sqrt{x^2-a^2}}$)

(i) $\int \frac{dx}{x\sqrt{x^2-2}}$. (ii) $\int \frac{dx}{x\sqrt{9x^2-1}}$.

(iii) $\int \frac{dx}{2x\sqrt{4x^2-1}}$. (iv) $\int \frac{dx}{\frac{x}{2}\sqrt{\frac{x^2}{4}-1}}$.

4. (Questions of the type $\int \sqrt{\frac{a+x}{a-x}}\, dx$)

(i) $\int \sqrt{\frac{1+x}{1-x}}\, dx$. (ii) $\int \sqrt{\frac{a-x}{a+x}}\, dx$.

(iii) $\int_0^a x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\, dx$. **(Hint.** Put $x^2=a^2 \cos 2\theta$)

5. (Questions of the type $\int \frac{f'(x)}{f(x)}\, dx$)

(i) $\int \frac{3x^2\, dx}{x^3+4}$. (ii) $\int \frac{3x+1}{3x^2+2x+1}\, dx$.

(iii) $\int \frac{dx}{(1+x^2)\tan^{-1} x}$. (iv) $\int \frac{\sec^2 x}{\tan x}\, dx$.

(v) $\int \frac{dx}{x \log x}$. (vi) $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\, dx$.

(vii) $\int \frac{1-\sin x}{x+\cos x}\, dx$. (viii) $\int \frac{\operatorname{cosec}^2 x\, dx}{\cot x}$.

(ix) $\int \frac{\sin x\, dx}{a+b\cos x}$. (x) $\int \frac{dx}{\sqrt{1-x^2}\sin^{-1} x}$.

(xi) $\int \frac{\cos(\tan^{-1} x)}{1+x^2}\, dx$. (xii) $\int \frac{\cos x\, dx}{1+\sin^2 x}$.

(xiii) $\int_0^{\pi/2} \frac{\cos x\, dx}{1+\sin x}$. (xiv) $\int \frac{dx}{x(1+\log x)}$.

(*K.U.* 1975)

(xv) $\int \frac{\sin x \cos x}{3\cos^2 x+2\sin^2 x}\, dx$.

6. (Questions of miscellaneous type)

(i) $\int \frac{e^{\tan^{-1} x}}{1+x^2}\, dx$. (ii) $\int e^x \cos e^x\, dx$.

(*K.U.* 1975)

(*iii*) $\int \frac{(\tan^{-1} x)^2}{1+x^2}\,dx.$ (*iv*) $\int \frac{dx}{x\,(\log x)^n}.$ (*v*) $\int \frac{(\log x)^n}{x}\,dx.$

(*vi*). $\int \sec^3 x \tan x\,dx.$ (*vii*) $\int \frac{3x^2\,dx}{1+x^6}.$ (Put $x^3=t$)

(*viii*) $\int \tan^5 x \sec^2 x\,dx.$ (*ix*) $\int \frac{\cot x}{\log \sin x}\,dx.$

(*x*) $\int \frac{\cos\theta\,d\theta}{1+\sin^2\theta}.$ (*xi*) $\int \frac{1}{x}\sin(\log x)\,dx.$

(*xii*) $\int \sin^3 x\,dx.$ (Put $\cos x=u$).

(*xiii*) $\int e^{\tan\theta}\sec^2\theta\,d\theta.$

(*xiv*) $\int \cos^5 x\,dx.$ (Put $\sin x=z$) (*K.U. 1977*)

(*xv*) $\int \frac{1}{\sqrt{x}}\cos\sqrt{x}\,dx.$

(*xvi*) $\int t^2(3+t^3)^{1/3}\,dt.$ (*K.U. 1965*)

(*xvii*) $\int 2t^2\sqrt{1+t^3}\,dt.$ (*K.U. 1964*)

(*xviii*) $\int \sin^3 x \cos^2 x\,dx.$ (Put $\cos x=t$).

(*xix*) $\int \sin x \cos^2 x\,dx.$ (*K.U. B.A. 1972*)

(*xx*) $\int \frac{z+1}{\sqrt{z^2+2z+1}}\,dz.$ (*K.U. Nov. 1975*)

8·12. Some Important Worked out Examples

We give below some important solved examples on different types of integration. These can be taken as articles, and the student is advised to understand them thoroughly.

Example 1. $\int \frac{dx}{a\sin x+b\cos x}.$ (*K.U. 1975*)

Sol. Put $a=r\cos\theta$ and $b=r\sin\theta$

so that $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\frac{b}{a}.$

$$\because \quad \int \frac{dx}{a\sin x+b\cos x}=\int \frac{dx}{r(\sin x\cos\theta+\cos x\sin\theta)}$$

$$=\frac{1}{r}\int \frac{dx}{\sin(x+\theta)}=\frac{1}{r}\int \operatorname{cosec}(x+\theta)\,dx$$

$$=\frac{1}{r}\log\tan\frac{x+\theta}{2}$$

$$=\frac{1}{\sqrt{a^2+b^2}}\log\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right).$$

Example 2. $\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$. (*K.U. 1977*)

Sol. Dividing the numerator and the denominator by $\cos^2 x$, we get

$$\int \frac{\frac{dx}{\cos^2 x}}{a^2 \frac{\sin^2 x}{\cos^2 x} + b^2} = \int \frac{\sec^2 x\, dx}{a^2 \tan^2 x + b^2}$$

Putting $a \tan x = z$, we have

$$a \sec^2 x\, dx = dz \text{ or } \sec^2 x\, dx = \frac{1}{a} dz$$

$$\therefore \int \frac{\sec^2 x\, dx}{a^2 \tan^2 x + b^2} = \frac{1}{a} \int \frac{dz}{z^2 + b^2} = \frac{1}{a} \cdot \frac{1}{b} \tan^{-1} \frac{z}{b}$$

$$= \frac{1}{ab} \tan^{-1} \left(\frac{a \tan x}{b} \right).$$

Example 3. $\int \frac{dx}{a^2 + \cos^2 x}$.

Sol. $$\int \frac{dx}{a^2 + \cos^2 x} = \int \frac{\frac{dx}{\cos^2 x}}{\frac{a^2}{\cos^2 x} + 1} = \int \frac{\sec^2 x\, dx}{a^2 \sec^2 x + 1}$$

$$= \int \frac{\sec^2 x\, dx}{a^2(1 + \tan^2 x) + 1} \quad \text{(Please note this step)}$$

$$= \frac{1}{a^2} \int \frac{\sec^2 x\, dx}{\tan^2 x + \left(1 + \frac{1}{a^2}\right)}$$

$$= \frac{1}{a^2} \int \frac{\sec^2 x\, dx}{\tan^2 x + \left(\sqrt{\frac{a^2+1}{a}}\right)^2}$$

Putting $\tan x = t$, we get $\sec^2 x\, dx = dt$

$$\therefore \frac{1}{a^2} \int \frac{\sec^2 x\, dx}{\tan^2 x + \left(\sqrt{\frac{a^2+1}{a}}\right)^2} = \frac{1}{a^2} \int \frac{dt}{t^2 + \left(\sqrt{\frac{a^2+1}{a}}\right)^2}$$

$$= \frac{1}{a^2} \cdot \frac{a}{\sqrt{a^2+1}} \tan^{-1} \left(\frac{t}{\sqrt{\frac{a^2+1}{a}}} \right) = \frac{1}{a\sqrt{a^2+1}} \tan^{-1} \left(\frac{a \tan x}{\sqrt{a^2+1}} \right).$$

Example 4. $\displaystyle\int \frac{dx}{(a \sin x+b \cos x)^2}$.

Sol. Dividing the numerator and the denominator by $\cos^2 x$, we get

$$\int \frac{dx}{(a \sin x+b \cos x)^2}=\int \frac{\dfrac{dx}{\cos^2 x}}{\left(a\dfrac{\sin x}{\cos x}+b\right)^2}$$

$$=\int \frac{\sec^2 x\, dx}{(a \tan x+b)^2}$$

Putting $\tan x=u$, we get

$$\sec^2 x\, dx=du$$

$$\therefore \quad \int \frac{\sec^2 x\, dx}{(a \tan x+b)^2}=\int \frac{du}{(au+b)^2}$$

$$=\int (au+b)^{-2}\, du=\frac{(au+b)^{-1}}{-a}=-\frac{1}{a}\,\frac{1}{au+b}$$

$$=-\frac{1}{a}\cdot\frac{1}{a \tan x+b}.$$

Example 5. $\displaystyle\int \frac{dx}{x\sqrt{x^4-1}}$. (*K.U. 1976*)

Sol. $\displaystyle\int \frac{dx}{x\sqrt{x^4-1}}=\int \frac{x\, dx}{x^2\sqrt{x^4-1}}$

Put $x^2=\sec\theta$, so that $x\, dx=\frac{1}{2}\sec\theta\tan\theta\, d\theta$

$$\therefore \quad \int \frac{x\, dx}{x^2\sqrt{x^4-1}}=\int \frac{\frac{1}{2}\sec\theta\tan\theta\, d\theta}{\sec\theta\sqrt{\sec^2\theta-1}}=\frac{1}{2}\int \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\tan\theta}$$

$$=\frac{1}{2}\int d\theta=\frac{1}{2}\theta=\frac{1}{2}\sec^{-1} x^2.$$

Exercise XX

Evaluate the following :

1. $\displaystyle\int \frac{dx}{\sin x+\cos x}$. 2. $\displaystyle\int \frac{dx}{3 \sin x+4 \cos x}$.

3. $\displaystyle\int \frac{dx}{5 \cos x-12 \sin x}$. 4. $\displaystyle\int \frac{\sin x\, dx}{\sin (x-a)}$. (*K.U. 1974*)

[**Hint :** $\sin x=\sin (x-a+a)$]

5. (*i*) $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}}$. (*ii*) $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$.

6. $\displaystyle\int \frac{dx}{1+3 \sin^2 x}$. 7. $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$.

8. $\int \frac{dx}{1+\cos^2 x}$. 9. $\int_0^{\pi/2} \frac{dx}{2+\sin^2 x}$. 10. $\int \frac{d\theta}{5+4\cos 2\theta}$.

11. $\int_0^{\pi/2} \frac{dx}{1+4\sin^2 x}$. 12. $\int \frac{dx}{(2\sin x+\cos x)^2}$.

13. $\frac{x^5 dx}{\sqrt{1+x^2}}$. *(K.U. 1977)* 14. $\int \sqrt{a^2-x^2}\, dx$.

15. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$. 16. $\int_0^{\pi} \frac{dx}{3+2\sin x+\cos x}$.

[**Hint.** Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and

$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$]

17. $\int \frac{\sin x\, dx}{\sqrt{1+\sin x}}$. 18. $\int \frac{\sin 2x\, dx}{(a+b\cos x)^2}$.

[**Hint.** Put $\cos x = t$]

19. $\int \sec x \log(\sec x+\tan x)\, dx$.

20. $\int_a^b \frac{\log x}{x} dx$. *(K.U., B.A., 1966)*

8·13. Integration by Partial Fraction

Sometimes the expression to be integrated can easily be resolved into *partial fractions*, which facilitates the integration. This is illustrated below by means of a few solved examples :

Example 1. $\int \frac{x-2}{(x+1)(x^2+1)} dx$.

Sol. Let $\frac{x-2}{(x+1)(x^2+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$,

$$x-2 = A(x^2+1)+(Bx+C)(x+1).$$

Putting $x=-1$, we get

$$2A=-3 \qquad \therefore \quad A=-\tfrac{3}{2}.$$

Also, equating the co-efficients of x^2 on both sides, we get

$$A+B=0 \text{ which gives } B=\tfrac{3}{2}.$$

Also, equating the coefficients of x, we get

$$B\mp C=1 \text{ which gives } c=-\tfrac{1}{2}.$$

$$\therefore \quad \frac{x-2}{(x+1)(x^2+1)} = \frac{-\frac{3}{2}}{x+1} + \frac{\frac{3}{2}x-\frac{1}{2}}{x^2+1}.$$

Hence

$$\int \frac{x-2}{(x+1)(x^2+1)}dx = -\tfrac{3}{2}\int \frac{dx}{x+1} + \tfrac{3}{2}\int \frac{x\,dx}{x^2+1} - \tfrac{1}{2}\int \frac{dx}{x^2+1}$$

$$= -\tfrac{3}{2}\int \frac{dx}{x+1} + \tfrac{3}{4}\int \frac{2x\,dx}{x^2+1} - \tfrac{1}{2}\int \frac{dx}{x^2+1}$$

$$= -\tfrac{3}{2}\log(x+1) + \tfrac{3}{4}\log(x^2+1) - \tfrac{1}{2}\tan^{-1}x.$$

Example 2. $\int \frac{dx}{x(x^n+1)}$. *(K.U., B.A. 1957)*

Sol. $\int \frac{dx}{x(x^n+1)} = \int \frac{x^{n-1}\,dx}{x^n(x^n+1)}$. (Please note this step)

Putting $x^n = z$, we get

$$x^{n-1}\,dx = \frac{1}{n}dz$$

$$\therefore \quad \int \frac{x^{n-1}\,dx}{x^n(x^n+1)} = \frac{1}{n}\int \frac{dz}{z(z+1)}$$

$$= \frac{1}{n}\int \left(\frac{1}{z} - \frac{1}{z+1}\right)dz$$

$$= \frac{1}{n}\int \frac{dz}{z} - \frac{1}{n}\int \frac{dz}{z+1}$$

$$= \frac{1}{n}\log z - \frac{1}{n}\log(z+1)$$

$$= \frac{1}{n}\log\frac{z}{z+1}$$

$$= \frac{1}{n}\log\frac{x^n}{x^n+1}.$$

Example 3. Show that

(*i*) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$ $(x > a)$.

(*ii*) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$.

Sol. (*i*) $\int \frac{dx}{x^2-a^2} = \int \frac{dx}{(x+a)(x-a)}$

$$= \frac{1}{2a}\int \left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}\{\log(x-a) - \log(x+a)\}$$

$$= \frac{1}{2a}\log\frac{x-a}{x+a}.$$

(ii) $\int \frac{dx}{a^2-x^2}=\int \frac{dx}{(a+x)(a-x)}=\frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)$

$$=\frac{1}{2a}[\log(a+x)-\log(a-x)]$$

$$=\frac{1}{2a}\log\frac{a+x}{a-x}.$$

Note. Example (3) is of great importance and can be treated as an article. The student is, therefore, required to commit it to memory along with its result.

Example 4. Evaluate $\int_0^{\pi/2}\frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$.

Sol. Put $\sin x=z$, so that $\cos x\,dx=dz$.

Also when $x=0$, $z=0$; when $x=\frac{\pi}{2}$, $z=1$.

$$\therefore \int_0^{\pi/2}\frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}=\int_0^1\frac{dz}{(1+z)(2+z)}$$

$$=\int_0^1\left(\frac{1}{z+1}-\frac{1}{z+2}\right)dz \qquad \text{(Please note this step)}$$

$$=\left|\log\frac{z+1}{z+2}\right|_0^1=\log\frac{2}{3}-\log\frac{1}{2}=\log\frac{4}{3}.$$

Example 5. Evaluate $\int\frac{dx}{4-5\sin^2 x}$. *(K.U. 1976)*

Sol. $\int\frac{dx}{4-5\sin^2 x}=\int\frac{\operatorname{cosec}^2 x\,dx}{4\operatorname{cosec}^2 x-5}$

$$=\int\frac{\operatorname{cosec}^2 x\,dx}{4(1+\cot^2 x)-5} \qquad \text{(Please note this step)}$$

$$=\int\frac{\operatorname{cosec}^2 x\,dx}{4\cot^2 x-1}.$$

Put $\cot x=t$ so that $\operatorname{cosec}^2 x\,dx=-dt$

$\therefore$ The integral $=\int\frac{dt}{1-4t^2}=\int\frac{dt}{(1+2t)(1-2t)}$

$$=\frac{1}{2}\int\left(\frac{1}{1+2t}+\frac{1}{1-2t}\right)dt$$

$$=\tfrac{1}{2}[\tfrac{1}{2}\log(1+2t)-\tfrac{1}{2}\log(1-2t)]$$

$$=\frac{1}{4}\log\frac{1+2t}{1-2t}=\frac{1}{4}\log\frac{1+2\tan x}{1-2\tan x}.$$

Example 6. Evaluate $\int \frac{dx}{(e^x-1)^2}$. *(K.U., B.A. 1977)*

Sol. Put $e^x-1=t$ so that $e^x\,dx=dt$

i.e., $$dx=\frac{dt}{e^x}=\frac{dt}{t+1}$$

$\therefore$ the integral $=\int \frac{dt}{t^2(t+1)}$

Now, let $\frac{1}{t^2(t+1)}=\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t+1}$

or $$1=At\,(t+1)+B(t+1)+Ct^2$$

Putting $t=0$, and $t=-1$, we get $B=1$ and $C=1$.

Again, equating co-efficients of t^2 from both sides, we get

$$A+C=1 \quad \text{or} \quad A=-C=-1$$

$$\therefore \quad \int \frac{dt}{t^2(t+1)}=\int\left(-\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t+1}\right)dt$$

$$=-\log t-\frac{1}{t}+\log(t+1)$$

$$=\log\frac{t+1}{t}-\frac{1}{t}=\log\frac{e^x}{e^x-1}-\frac{1}{e^x-1}$$

Exercise XXI

Evaluate the following :

1. $\int \frac{dx}{x^2-4}$.
2. $\int \frac{dx}{6x^2+5x-4}$.
3. $\int \frac{x^2\,dx}{(x+1)^2(x+2)}$.
4. $\int \frac{(x^2+6)\,dx}{(x^2+4)(x^2+9)}$.
5. $\int \frac{x^2+1}{x(x^2-1)}\,dx$.
6. $\int \frac{x^2\,dx}{1-x^6}$.
7. $\int \frac{2x^2\,dx}{(x-1)(x^2+1)}$
8. $\int \frac{3x+1}{(x-1)^2(x+3)}\,dx$.
9. $\int \frac{2x-5}{x^3+5x^2+7x+3}\,dx$.
10. $\int \frac{x^2\,dx}{(x+a)(x+b)(x+c)}$
11. $\int \frac{dx}{e^x+e^{2x}}$.
12. $\int_0^\infty \text{sech}\,x\,dx$.
13. $\int \frac{dx}{\sin x+\sin 2x}$.

8·13. (*a*) **Integration by parts**

Rule. This rule follows from the rule for the differentiation of a product of two functions. If u and v be two functions, then

$$d(uv)=udv+vdu$$

or
$$udv=d(uv)-vdu$$

Integrating both sides, we get

$$\int \mathbf{udv}=\mathbf{uv}-\int \mathbf{vdu}$$

which gives the rule.

Example 1. Evaluate : $\int xe^x\,dx$.

Sol. Take $u=x$ and $v=e^x$

$\therefore \quad \int xe^x\,dx=xe^x-\int 1.e^x\,dx=xe^x-e^x.$

Note. Proper choice of u and v is essential as a wrong choice of these may make the integral complicated.

Example 2. Evaluate : $\int \cos x \log \sin x\,dx$. (*K.U., B.A. 1966*)

Sol. Take $u=\cos x$ and $v=\log \sin x$

$$\therefore \quad \int \cos x \log \sin x\,dx=\sin x \log \sin x-\int \sin x \,.\, \frac{\cos x}{\sin x}\,dx$$

$$=\sin x \log \sin x-\sin x.$$

Note. *Inverse trigonometric functions and lagarithmic functions should never be taken as u.*

Example 3. Evaluate : $\int \tan^{-1} x\,dx$.

Sol. $\int \tan^{-1} x\,dx=\int 1\,.\,\tan^{-1} x\,dx$ (Please note this step)

Take $u=1$ and $v=\tan^{-1} x$

$$\therefore \quad \int \tan^{-1} x\,dx=x\tan^{-1} x-\int x\,.\,\frac{1}{1+x^2}\,dx$$

$$=x\tan^{-1} x-\frac{1}{2}\int \frac{2x\,dx}{1+x^2}$$

$$=x\tan^{-1} x-\tfrac{1}{2}\log(1+x^2).$$

Example 4. Evaluate $\displaystyle\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx.$

(*K.U., B.A. 1972*)

Sol. Put $x=\cos\theta$ so that $dx=-\sin\theta\,d\theta$.

$$\therefore \quad \text{The integral}=\int \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,(-\sin\theta)\,d\theta$$

$$=\int \tan^{-1}\left(\tan\frac{\theta}{2}\right)(-\sin\theta)\,d\theta$$

$$=-\tfrac{1}{2}\int \theta\sin\theta\,d\theta$$

$$=-\tfrac{1}{2}\,[\theta\,(-\cos\theta)-\int 1\,.\,(-\cos\theta)\,d\theta]$$

(Integrating by parts)

$$=\tfrac{1}{2}\,\theta\cos\theta-\tfrac{1}{2}\int\cos\theta\,d\theta$$

$$=\tfrac{1}{2}\,\theta\cos\theta-\tfrac{1}{2}\sin\theta$$

$$=\tfrac{1}{2}\,.\,x\cos^{-1}x-\tfrac{1}{2}\sqrt{1-x^2}.$$

$(\because\ \theta=\cos^{-1}x$ and $\sin\theta=\sqrt{1-x^2})$

Example 5. Evaluate $\int\sin^{-1}\sqrt{\dfrac{x}{a+x}}\,dx.$ (*K.U. 1976*)

Sol. Put $x=a\tan^2\theta$, so that

$$dx=2a\tan\theta\sec^2\theta\,d\theta.$$

$\therefore$ the integral $=\displaystyle\int\sin^{-1}\sqrt{\frac{a\tan^2\theta}{a(1+\tan^2\theta)}}\times 2a\tan\theta\sec^2\theta\,d\theta$

$$=\int\sin^{-1}\sin\theta\times 2a\tan\theta\sec^2\theta\,d\theta$$

$$=2a\int\theta\tan\theta\sec^2\theta\,d\theta$$

Integrating by parts, taking $u=\tan\theta\sec^2\theta$ and $v=\theta$, we get

the integral $=2a\left[\dfrac{\theta\tan^2\theta}{2}-\displaystyle\int 1\,.\,\frac{\tan^2\theta}{2}\,d\theta\right]$

$$\left(\because\ \int\tan\theta\sec^2\theta\,d\theta=\frac{\tan^2\theta}{2}\right)$$

$$=a(\theta\tan^2\theta-\int\tan^2\theta\,d\theta)$$

$$=a[\theta\tan^2\theta-\int(\sec^2\theta-1)\,d\theta]$$

$$=a[\theta\tan^2\theta-\tan\theta+\theta]$$

$$=a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}+\tan^{-1}\sqrt{\frac{x}{a}}\right]$$

$$=(x+a)\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{ax}.$$

Example 6. Evaluate $\int e^x\,\dfrac{1+x\log x}{x}\,dx.$ (*K.U. 1976*)

Sol. The integral $=\displaystyle\int\frac{1}{x}\,e^x\,dx+\int e^x\log x\,dx.$

Let $\quad I=\displaystyle\int e^x\log x\,dx=e^x\log x-\int e^x\,.\,\frac{1}{x}\,dx$

(Integrating by parts)

$\therefore$ The integral $=\displaystyle\int\frac{1}{x}\,e^x\,dx+e^x\log x-\int\frac{1}{x}\,e^x\,dx$

(Substituting for $\int e^x\log x$)

$$=e^x\log x.$$

Example 7. Evaluate $\int \frac{xe^x}{(x+1)^2}\,dx.$ *(K.U. 1977)*

Sol. The integral $=\int \frac{(x+1)\,e^x-e^x}{(x+1)^2}\,dx$

$$=\int \frac{e^x\,dx}{(x+1)}-\int \frac{e^x\,dx}{(x+1)^2}.$$

Let $$I=\int \frac{e^x\,dx}{(x+1)}$$

$\therefore$ $$I=\frac{e^x}{x+1}-\int e^x\left[-\frac{1}{(x+1)^2}\right]dx$$

(Integrating by parts)

$\therefore$ The integral $$=\frac{e^x}{x+1}+\int \frac{e^x\,dx}{(x+1)^2}-\int \frac{e^x\,dx}{(x+1)^2}$$

$$=\frac{e^x}{x+1}.$$

Example 8. Integrate $\sqrt{a^2-x^2}$ by parts with respect to x. *(K.U., B.A. 1957)*

Sol. Let $$I=\int \sqrt{a^2-x^2}\,dx=\int 1\,.\,\sqrt{a^2-x^2}\,dx$$

$$=x\sqrt{a^2-x^2}-\int \frac{x(-x)}{\sqrt{a^2-x^2}}\,dx$$

(Integrating by parts)

or $$I=x\sqrt{a^2-x^2}-\int \frac{-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$=x\sqrt{a^2-x^2}-\int \frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx$$

(Please mark this step)

$$=x\sqrt{a^2-x^2}-\int \sqrt{a^2-x^2}\,dx+a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$=x\sqrt{a^2-x^2}-I+a^2\sin^{-1}\frac{x}{a} \qquad \text{(Art. 8·11)}$$

or $$2I=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}$$

$\therefore$ $$I=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

Example 9. Evaluate $\int e^{ax}\sin(bx+c)\,dx.$

Sol. Let $I=\int e^{ax}\sin(bx+c)\,dx.$

Take $u=e^{ax}$, $v=\sin(bx+c)$, and then integrating by parts, we get

$$I=\frac{e^{ax}}{a}\sin(bx+c)-\int \frac{e^{ax}}{a}\times b\cos(bx+c)\,dx$$

$$=\frac{e^{ax}}{a}\sin(bx+c)-\frac{b}{a}\int e^{ax}\cos(bx+c)\,dx$$

$$=\frac{e^{ax}}{a}\sin(bx+c)-\frac{b}{a}\left[\frac{e^{ax}}{a}\cos(bx+c)\right.$$

$$\left.-\int\frac{e^{ax}}{a}\times -b\sin(bx+c)\,dx\right]$$

[Integrating $e^{ax}\cos(bx+c)$ also by parts]

or $$=\frac{e^{ax}}{a}\sin(bx+c)-\frac{b}{a^2}\,e^{ax}\cos(bx+c)$$

$$-\frac{b^2}{a^2}\int e^{ax}\sin(bx+c)dx$$

or $$\mathrm{I}=\frac{e^{ax}}{a}\sin(bx+c)-\frac{b}{a^2}\,e^{ax}\cos(bx+c)-\frac{b^2}{a^2}\,\mathrm{I}$$

[$\because\ \mathrm{I}=\int e^{ax}\sin(bx+c)\,dx$]

or $$\left(1-\frac{b^2}{a^2}\right)\mathrm{I}=\frac{e^{ax}}{a}\sin(bx+c)-\frac{b}{a^2}\,e^{ax}\cos(bx+c)$$

or $$\frac{(a^2+b^2)\,\mathrm{I}}{a^2}=\frac{e^{ax}}{a^2}\{a\sin(bx+c)-b\cos(bx+c)\}$$

or $$\mathrm{I}=\frac{e^{ax}\{a\sin(bx+c)-b\cos(bx+c)\}}{a^2+b^2}.$$

This result can be put in another form by

$a=r\cos\theta,\quad b=r\sin\theta$ so that

$r=\sqrt{a^2+b^2},\quad \theta=\tan^{-1}\dfrac{b}{a}.$

$$\therefore\quad \mathrm{I}=\frac{e^{ax}\,.\,r}{a^2+b^2}\{\sin(bx+c)\cos\theta-\cos(bx+c)\sin\theta\}$$

$$=\frac{e^{ax}\sqrt{a^2+b^2}}{a^2+b^2}\sin(bx+c-\theta)$$

$$=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx+c-\tan^{-1}\frac{b}{a}\right).$$

Example 10. Show that

$$\int e^{ax}\cos(bx+c)\,dx=\frac{e^{ax}\cos\left(bx+c-\tan^{-1}\dfrac{b}{a}\right)}{\sqrt{a^2+b^2}}$$

This solution is left as an exercise for the student as he has to proceed in exactly the same way as in the case of example 9.

Note. *Examples (9) and (10) can be regarded as articles of immense importance, and the student is, therefore, required to memorize their solutions.*

Example 11. Evaluate $\int \cos 2\theta \log \frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta} d\theta.$

Sol. Take $u=\cos 2\theta$, $\quad v=\log \frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta}.$

$\therefore$ The integral

$$=\frac{\sin 2\theta}{2} \log \frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta}-\int \frac{\sin 2\theta}{2} \cdot \frac{d}{d\theta}\left\{ \log \frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta}\right\} d\theta$$

$$=\frac{\sin 2\theta}{2} \log \frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta}$$

$$-\int \frac{\sin 2\theta}{2}\left\{\frac{-\sin \theta+\cos \theta}{\cos \theta+\sin \theta}-\frac{-\sin \theta-\cos \theta}{\cos \theta-\sin \theta}\right\} d\theta$$

$$=\frac{\sin 2\theta}{2} \log \frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta}$$

$$-\int \frac{\sin 2\theta}{2}\left\{\frac{(\cos \theta-\sin \theta)^2+(\cos \theta+\sin \theta)^2}{\cos^2 \theta-\sin^2 \theta}\right\} d\theta$$

$$=\frac{\sin 2\theta}{2} \log \frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta}-\int \frac{\sin 2\theta}{2} \cdot \frac{2\,(\cos^2 \theta+\sin^2 \theta)}{\cos^2 \theta-\sin^2 \theta} d\theta$$

$$=\frac{\sin 2\theta}{2} \log \frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta}-\int \frac{\sin 2\theta}{\cos 2\theta} d\theta$$

$$=\frac{\sin 2\theta}{2} \log \frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta}+\frac{1}{2} \log \cos 2\theta.$$

Example 12. Evaluate $\int \log (1+x^2)\, dx$.

Sol. The integral $=\int 1 \,.\, \log (1+x^2)\, dx$ (Please note this step)

Take $u=1$, $\quad v=\log (1+x^2)$

$\therefore$ Integrating by parts, we get

$$\int 1 \,.\, \log (1+x^2)\, dx=x \log (1+x^2)-\int x \cdot \frac{2x}{1+x^2} dx$$

$$=x \log (1+x^2)-2 \int \frac{x^2\, dx}{1+x^2}$$

$$=x \log (1+x^2)-2 \int\left(1-\frac{1}{1+x^2}\right) dx$$

(Please note this step)

$$=x \log (1+x^2)-2 \int dx+2 \int \frac{dx}{1+x^2}$$

$$=x \log (1+x^2)-2x+2 \tan^{-1} x.$$

Exercise XXII

Integrate the following by parts :

1. $x^3 e^x$. 2. $x \log x$. 3. $x \sin x$.
4. $x \sec^2 x$. 5. $x \sec x \tan x$. 6. $\log x$.
7. $x^2 \sin^{-1} x$. 8. $x^3 \tan^{-1} x$. 9. $x \sin x \cos x$.
10. $\sin x \log \cos x$. 11. $(\log x)^2$.
12. $\sin^{-1} x$. 13. $x \sinh x$. 14. $\sqrt{a^2+x^2}$.
15. $\sqrt{x^2-a^2}$. 16. $e^{3x} \sin 4x$.
17. $e^x \tan^{-1} (e^x)$. (*K.U. Nov. 1975*)
18. $\cot^4 x$. (*K.U. Nov. 1975*)

[**Hint.** $\cot^4 x=(\operatorname{cosec}^2 x-1) \cot^2 x$, etc.]

19. $\dfrac{e^{a \tan^{-1} x}}{1+x^2}$. 20. $\dfrac{x \sin^{-1} x}{\sqrt{1-x^2}}$. (*K.U. 1977*)

21. $e^x \dfrac{1-\sin x}{1-\cos x}$. 22. $\operatorname{cosec}^3 x$. (*K.U. 1977*)

23. $e^x (x \sin x+\cos x)$. (*K.U. 1976*)

Show that :

24. $\displaystyle\int \cos (\log x)\, dx=\frac{1}{\sqrt{2}}\, x \cos \left(\log x-\frac{\pi}{4} \right)$.

25. $\displaystyle\int x^3 \sin (a \log x)\, dx=\frac{x^4}{\sqrt{16+a^2}} \sin \left(a \log x-\tan^{-1} \frac{a}{4} \right)$.

[**Hint.** Put $x=e^z$.]

26. $\displaystyle\int x \sin^{-1} \frac{1}{2} \sqrt{\frac{2a-x}{a}}\, dx=a^2 \left[\frac{\theta \cos 2\theta}{2}-\frac{\sin 2\theta}{4}\right]$

where $x=2a \cos \theta$.

27. $\displaystyle\int \cos \left(b \log \frac{x}{a} \right) dx=\frac{x}{\sqrt{b^2+1}} \cos \left(b \log \frac{x}{a}-\tan^{-1} b \right)$.

$\left[\textbf{Hint.}\ \text{Put } \dfrac{x}{a}=e^z\right]$

28. $\displaystyle\int 2^x \sin 2x\, dx=\frac{1}{\sqrt{1+(\log 2)^2}}.2^x.\sin \left(2x-\tan^{-1} \frac{2}{\log 2}\right)$.

[**Hint.** $2^x=e^{x \log 2}$, etc.]

29. $\displaystyle\int \cos^{-1} \frac{1}{x}\, dx=x \cos^{-1} \frac{1}{x}-\log (x+\sqrt{x^2-1})$.

30. If $u=\int e^{ax} \cos bx\, dx$ and $v=\int e^{ax} \sin bx\, dx$, prove that

(i) $\tan^{-1} \dfrac{v}{u} + \tan^{-1} \dfrac{b}{a} = bx.$

(ii) $(a^2+b^2)(u^2+v^2)=e^{2ax}.$

8·14. Integration as the limit of a sum

So far we have all along regarded integration as the inverse of differentiation. However, it will be in the present article that a *definite integral* can also be represented as the *limit of the sum of a certain number of terms* when the *number* of such terms tends to infinity and each term tends to zero.

8·15. Fundamental Theorem of Integral Calculus

Let the interval (a, b) *be divided into* n *equal parts and let the length of each part be* h, *so that* $b-a=nh$, *then*

$$\int_a^b f(x) = \text{Lt } h\,[f(a)+f(a+h)+f(a+2h)+\ldots\ldots+f(a+\overline{n-1}\,h)]$$

when $\quad n \to \infty, \quad h \to 0 \quad$ and $\quad b-a=nh.$

Note. The proof of this theorem is beyond the scope of this book.

With the help of this theorem, we are now in a position to perform integration by regarding it as the limit of the summation of the series in which the number of terms becomes infinite. The method is explained below with the help of a few solved examples :

Example 1. Evaluate $\int_a^b x^2\, dx$ as the limit of the sum.

(*K.U. 1977*)

Sol. Here $f(x)=x^2$

$$\therefore \int_a^b f(x)\, dx = \text{Lt } h\,[a^2+(a+h)^2+(a+2h)^2+\ldots\ldots$$
$$\ldots\ldots+(a+\overline{n-1}\,h)^2]$$

Where $n \to \infty$, $h \to 0$, and $b-a=nh$

$$=\text{Lt } h[na^2+2ah\,(1+2+\ldots\ldots+\overline{n-1})+h^2\,\{1^2+2^2+\ldots\ldots$$
$$\ldots\ldots+(n-1)^2\}]$$

$$=\text{Lt } h\left[na^2+2ah\,\frac{n(n-1)}{2}+h^2\,\frac{(n-1)\,n(2n-1)}{6}\right]$$

$$=\text{Lt}\left[nha^2+a.nh(nh-h)+\frac{(nh-h).nh.(2nh-h)}{6}\right]$$

(Please note this step)

$$=\text{Lt}\left[(b-a)\,a^2+a(b-a)(b-a-h)+\frac{(b-a-h)(b-a)(2b-2a-h)}{6}\right]$$

$$=(b-a)a^2+a(b-a)^2+\frac{(b-a)^3}{3} \qquad (\because \quad h\to 0)$$

$$=\frac{b^3-a^3}{3}.$$

Example 2. Evaluate $\int_a^b e^x\,dx$ as the limit of the sum.

Sol. Here $f(x)=e^x$

$$\therefore \quad \int_a^b e^x\,dx=\text{Lt}\,h\left[e^a+e^{a+h}+e^{a+2h}+\ldots\ldots+e^{a+\overline{n-1}\,h}\right]$$

when $n\to\infty$, $h\to 0$, $nh=b-a$

$$=\text{Lt}\,h\left[\frac{e^a(e^{nh}-1)}{e^h-1}\right] \qquad (\because \quad e^h>1)$$

$$=\underset{h\to 0}{\text{Lt}}\frac{e^a(e^{nh}-1)}{\frac{e^h-1}{h}} \qquad \text{(Please not this step)}$$

$$=\frac{e^a(e^{b-a}-1)}{\underset{h\to 0}{\text{Lt}}\frac{e^h-1}{h}} \qquad (\because \quad nh=b-a)$$

$$=e^a\,(e^{b-a}-1)=e^b-e^a. \qquad \left(\because \underset{h\to 0}{\text{Lt}}\frac{e^h-1}{h}=1\right)$$

Example 3. Evaluate : $\int_a^b \cos x\,dx$ as the limit of a sum.

Sol. Here $f(x)=\cos x$

$$\therefore \quad \int_a^b \cos x\,dx=\text{Lt}\,h\,[\cos a+\cos(a+h)+\cos(a+2h)+\ldots\ldots+\cos(a+\overline{n-1}\,h)] \quad \ldots(1)$$

Now let

$$S=\cos a+\cos(a+h)+\cos(a+2h)+\ldots\ldots+\cos(a+\overline{n-1}\,h).$$

Multiplying both sides by $2 \sin \frac{h}{2}$, we get

$$2 \sin \frac{h}{2}\ S=2 \sin \frac{h}{2} \cos a+2 \sin \frac{h}{2} \cos (a+h)+\ldots$$
$$\ldots+\sin \frac{h}{2} \cos (a+\overline{n-1}\ h)$$
$$=\sin\left(a+\frac{h}{2}\right)-\sin\left(a-\frac{h}{2}\right)+\sin\left(a+\frac{3h}{2}\right)-\sin\left(a+\frac{h}{2}\right)$$
$$+\ldots\ldots+\sin\left(a+\overline{2n-1}\ \frac{h}{2}\right)-\sin\left(a+\overline{2n-3}\frac{h}{2}\right)$$

or $$S=\frac{\sin\left(a+\overline{2n-1}\ \frac{h}{2}\right)-\sin\left(a-\frac{h}{2}\right)}{2 \sin \frac{h}{2}}.$$

This gives (1) as

$$\int_a^b \cos x\, dx=\underset{h\to 0}{\mathrm{Lt}}\left[\frac{\sin\left(a+\overline{2n-1}\ \frac{h}{2}\right)-\sin\left(a-\frac{h}{2}\right)}{\sin \frac{h}{2} \Big/ \frac{h}{2}}\right]$$

(Please note this step)

$$=\underset{h\to 0}{\mathrm{Lt}}\left[\frac{\sin\left(a+nh-\frac{h}{2}\right)-\sin\left(a-\frac{h}{2}\right)}{\sin \frac{h}{2} \Big/ \frac{h}{2}}\right]$$

$$=\sin (a+b-a)-\sin a$$

$$\left(\because \underset{h\to 0}{\mathrm{Lt}} \frac{\sin \frac{h}{2}}{\frac{h}{2}}=1 \text{ and } h\to 0\right)$$

$$=\sin b-\sin a.$$

Note. Integration *as the limit of a sum* is also known as the integration *from first principles* or the integration *ab-initio.*

Exercise XXIII

Integrate the following as limits of sums :

1. $\int_a^b x\, dx.$ **2.** $\int_a^b m^x\, dx.$ **3.** $\int_a^b x^2\, dx.$

(K.U. 1977)

4. $\int_a^b \sin x\, dx.$ **5.** $\int_a^b \cos^2 x\, dx.$ **6.** $\int_a^b \sin^2 x\, dx.$

7. $\int_a^b x^3\, dx.$ **8.** $\int_0^{\pi/2} \cos x\, dx.$

$\left[\textbf{Hint.} \text{ In the answer put } a=0,\ b=\frac{\pi}{2}\right]$

9. $\int x\, dx.$ [**Hint.** In the answer put $a=0$, $b=x$]

10. $\int_a^b e^{3x}\, dx.$

9

Statement of Maclaurin's Theorem

9·1. *Statement.* Let $f(x)$ be a function of x which can be expanded in ascending powers of x. Let us assume further that the expansion is differentiable term by term any number of times, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$$
$$\ldots\ldots+\frac{x^n}{n!}f^n(0)+\ldots$$

Proof.* Suppose

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots\ldots \quad \ldots(1)$$

Then by successive differentiation, we get

$$f'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots\ldots \quad \ldots(2)$$
$$f''(x)=2a_2+2.3.a_3x+3.4.a_4x^2+\ldots\ldots \quad \ldots(3)$$
$$f'''(x)=2.3.a_3+2.3.4.a_4x+\ldots\ldots \quad \ldots(4)$$

Putting $x=0$ in (1), (2), (3), (4), we get

$f(0)=a_0$, $f'(0)=a_1$, $f''(0)=a_2\,(2\,!)$,
$f'''(0)=a_3\,(3\,!)$......and so on.

Substituting the values of a_0, a_1, a_2, a_3 ...... in the R.H.S. of (1), we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

9·2. Application to Simple Expansion

The method has been explained with the help of the following worked out examples :

Example 1. Expand (*i*) e^x, and (*ii*) $\log(1+x)$ with the help of Maclaurin's Theorem.

*This proof is by no means a rigorous one. No proof for this theorem has, however, been prescribed in the University syllabus.

Sol. (*i*) Let $f(x)=e^x$, then $f(0)=e^0=1$

$$f'(x)=e^x \qquad f'(0)=e^0=1$$
$$f''(x)=e^x \qquad f''(0)=e^0=1$$
$$f'''(x)=e^x \qquad f'''(0)=e^0=1$$

and so on.

Hence

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots$$

gives
$$e^x=1+x.1+\frac{x^2}{2!}.1+\frac{x^3}{3!}.1+\ldots\ldots\ldots$$
$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots$$

(*ii*) Here $f(x)=\log(1+x)$ $\therefore$ $f(0)=\log 1=0$

$$f'(x)=(1+x)^{-1} \qquad f'(0)=1$$
$$f''(x)=-(1+x)^{-2} \qquad f''(0)=-1$$
$$f'''(x)=2(1+x)^{-3} \qquad f'''(0)=2$$
$$f''''(x)=-6(1+x)^{-4} \qquad f''''(0)=-6$$

and so on.

Hence

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots\ldots$$

gives

$$\log(1+x)=0+x(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2)+\frac{x^4}{4!}(-6)+\ldots\ldots$$
$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots$$

Example 2. Show that :

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\ldots$$

Sol. Here $f(x)=\cos x$ $\therefore$ $f(0)=\cos 0=1$

Now, $f^n(x)=\cos\left(x+\frac{n\pi}{2}\right)$ [Art. 5·2 (5)]

$\therefore$ $f^n(0)=\cos\frac{n\pi}{2}$

Putting $n=1, 2, 3, 4, 5$, we get

$$f'(0)=\cos\frac{\pi}{2}=0$$
$$f''(0)=\cos\pi=-1$$

$$f'''(0)=\cos\frac{3\pi}{2}=\cos\left(\pi+\frac{\pi}{2}\right)=-\cos\frac{\pi}{2}=0$$

$$f''''(0)=\cos 2\pi=1$$

$$f''''' (0)=\cos\frac{5\pi}{2}=\cos\left(2\pi+\frac{\pi}{2}\right)=\cos\frac{\pi}{2}=0$$

and so on.

$$\therefore \qquad f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots$$

gives
$$\cos x=1+x(0)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(1)+\frac{x^5}{5!}+\ldots\ldots$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\ldots$$

Example 3. Apply Maclaurin's Theorem to expand $\tan x$ as far as the term in x^5. *(K.U. 1977)*

Sol. Here $f(x)=\tan x \qquad \therefore \qquad f(0)=0$

$$f'(x)=\sec^2 x \qquad \therefore \qquad f'(0)=1$$

$$f''(x)=2\sec^2 x\tan x \qquad \therefore \qquad f''(0)=0$$

$$f'''(x)=4\sec^2 x\tan^2 x+2\sec^4 x$$

$$\therefore \qquad f'''(0)=2$$

$$f''''(x)=16\sec^4 x\tan x+8\sec^2 x\tan^2 x$$

$$\therefore \qquad f''''(0)=0$$

$$f'''''(x)=16\sec^6 x+88\sec^4 x\tan^2 x+16\sec^2 x\tan^4 x$$

$$\therefore \qquad f'''''(0)=16$$

and so on.

Hence $f(x)=f(0)+xf'(0)+\dfrac{x^2}{2!}f''(x)+\ldots\ldots$

gives
$$\tan x=0+x(1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(2)+\frac{x^4}{4!}(0)+\frac{x^5}{5!}(16)+\ldots\ldots$$

$$=x+\frac{x^3}{2}+\frac{2x^5}{15}+\ldots\ldots$$

Example 4. Obtain the expansion of $e^{a\sin^{-1}x}$. *(K.U. 1976)*

Sol. Here $f(x)=e^{a\sin^{-1}x}$ ...(1)

$\therefore \qquad f(0)=e^0=1$

Let denote $f(x)$ by y, so that

$$f(0)=(y)_0$$

$$f'(0)=(y_1)_0, \qquad f''(0)=(y_2)_0, \qquad f'''(0)=(y_3)_0$$

and so on.

Now, (1) gives ·

$$y=e^{a \sin^{-1} x}$$

$$\therefore \qquad y_1=\frac{a}{\sqrt{1-x^2}}\ e^{a \sin^{-1} x}=\frac{ay}{\sqrt{1-x^2}} \qquad \ldots(2)$$

(Please note this step)

or $$(y_1)_0=\frac{a\,(y)_0}{1}=\frac{a\,.\,1}{1}=a.$$

Also (2) gives

$$(1-x^2)\, y_1^2=a^2y^2.$$

Differentiating again and removing the common factor $2y_1$, we get

$$(1-x^2)\, y_2-xy_1-a^2y=0 \qquad \ldots(3)$$

Putting $x=0$ in (3), we get

$$(y_2)_0=a^2\,(y)_0=a^2\,.\,1=a^2.$$

Again, differentiating (3) n times by Leibnitz's Theorem, we have

$$(1-x^2)\, y_{n+2}-(2n+1)\,xy_{n+1}-(n^2+a^2)\, y_n=0$$

[See Ex. (2) after Art. 5·4]

Putting $x=0$, we get

$$(y_{n+2})_0=(n^2+a^2)(y_n)_0$$

Hence, putting $n=1, 2, 3, \ldots\ldots$, we get

$$(y_3)_0=a(1^2+a^2)$$

$$(y_4)_0=a^2(2^2+a^2)$$

$$(y_5)_0=(3^2+a^2)(y_3)_0=a(1^2+a^2)(3^2+a^2)$$

and so on.

Hence $$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\ldots\ldots$$

gives $$e^{a \sin^{-1} x}=1+ax+\frac{a^2}{2\,!}x^2+\frac{a(1^2+a^2)}{3\,!}x^3+\frac{a^2(2^2+a^2)}{4\,!}x^4+\ldots\ldots$$

Exercise XXIV

Apply Maclaurin's Theorem to show that :

1. $e^{-x}=1-x+\dfrac{x^2}{2\,!}-\dfrac{x^3}{3\,!}+\dfrac{x^4}{4\,!}-\ldots\ldots$

2. $a^{mx}=1+(m\log a)\,x+\dfrac{(m\log a)^2}{2\,!}x^2+\ldots\ldots$

3. (i) $\sin x=x-\dfrac{x^3}{3\,!}+\dfrac{x^5}{5\,!}\ldots\ldots\ldots$

   (ii) $\cos x=1-\dfrac{x^2}{2\,!}+\dfrac{x^4}{4\,!}\ldots\ldots\ldots$

   (iii) $\tan x=x+\dfrac{x^3}{3}+\dfrac{2}{15}x^5+\ldots\ldots$

4. $\tan^{-1}x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots\ldots$

5. $e^{\sin x}=1+x+\dfrac{x^2}{2}+\dfrac{x^4}{8}+\ldots\ldots$

6. $\log\sec x=\frac{1}{2}x^2+\dfrac{x^4}{12}+\dfrac{x^6}{45}+\ldots\ldots$

7. $\log(1-x)=-x-\dfrac{x^2}{2}-\dfrac{x^3}{2}-\dfrac{x^4}{4}-\ldots\ldots$

8. $\log(1+e^x)=\log 2+\frac{1}{2}x+\frac{1}{4}\cdot\dfrac{x^2}{2\,!}+\ldots\ldots$

9. $\sec x=1+\dfrac{x^2}{2\,!}+\dfrac{5x^4}{4\,!}+\ldots\ldots$

10. (i) $\sinh x=x+\dfrac{x^3}{3\,!}+\dfrac{x^5}{5\,!}+\ldots\ldots$

    (ii) $\cosh x=1+\dfrac{x^2}{2\,!}+\dfrac{x^4}{4\,!}+\ldots\ldots$

    [**Hint.** $\sinh x=\dfrac{e^x-e^{-x}}{2}$ etc.]

11. $\log(1+\tan x)=x-\dfrac{x^2}{2}+\frac{2}{3}x^3-\ldots\ldots$

12. $\log(1+\sin x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{6}-\dfrac{x^4}{12}+\ldots\ldots$

13. $\sin^{-1}x=x+\frac{1}{2}\cdot\dfrac{x^3}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\dfrac{x^5}{5}+\ldots\ldots$

14. $\log(1+x+x^3)=x+\dfrac{x^2}{2}-\dfrac{2x^3}{3}+\dfrac{x^4}{4}-\ldots\ldots$

    [**Hint.** $\log(1+x+x^2)=\log\dfrac{1-x^3}{1-x}$ etc.]

15. $\dfrac{e^x}{\cos x}=1+x+\dfrac{2x^2}{2\,!}+\dfrac{4x^3}{3\,!}+\ldots\ldots$

16. $e^{x\cos x}=1+x+\dfrac{x^2}{2\,!}-\dfrac{x^3}{3\,!}+\ldots\ldots$

17. $e^{ax}\cos bx=1+ax+(a^2-b^2)\,\dfrac{x^2}{2\,!}+\dfrac{a(a^2-3b^2)}{3\,!}x^3+\ldots\ldots$

18. $\sin(m\sin^{-1}x)=mx+\dfrac{m(1^2-m^2)}{3\,!}\,x^3+\dfrac{m(1^2-m^2)(3^2-m^2)}{5\,!}x^5+\ldots\ldots$

19. $(1+x)^{1+x}=1+x+x^2+\dfrac{x^3}{3}+\ldots\ldots$

20. $\sin(e^x-1)=x+\dfrac{x^2}{2}-\dfrac{5}{24}x^4+\ldots\ldots$

21. $\log\{1-\log(1-x)\}=x+\dfrac{x^3}{6}-\ldots\ldots$

22. $e^{m\tan^{-1}x}=1+mx+\dfrac{m^2}{2\,!}x^2+\dfrac{m(m^2-2)}{3\,!}x^3+\ldots\ldots$

23. $\dfrac{e^x}{e^x+1}=x+\dfrac{x^3}{3}+\ldots\ldots$

9·3. Evaluation of some limits with the help of expansion

The method is explained below by means of some solved examples. It is not, however, always practicable to apply the method.

Example 1. Evaluate :

$$\underset{\theta\to 0}{\mathrm{Lt}}\ \frac{\sin\theta}{\theta}.$$

Sol.
$$\underset{\theta\to 0}{\mathrm{Lt}}\ \frac{\sin\theta}{\theta}=\underset{\theta\to 0}{\mathrm{Lt}}\ \frac{\theta-\dfrac{\theta^3}{3\,!}+\dfrac{\theta^5}{5\,!}-\ldots\ldots}{\theta}$$

(Expanding $\sin\theta$)

$$=\underset{\theta\to 0}{\mathrm{Lt}}\left(1-\frac{\theta^2}{3\,!}+\frac{\theta^4}{5\,!}-\ldots\ldots\right)=1.$$

Example 2. Evaluate :

$$\underset{x\to 0}{\mathrm{Lt}}\ \frac{a^x-1}{x}.$$

Sol.
$$\underset{x\to 0}{\mathrm{Lt}}\ \frac{a^x-1}{x}=\underset{x\to 0}{\mathrm{Lt}}\ \frac{1+x(\log a)+\dfrac{x^2}{2\,!}(\log a)^2+\ldots.-1}{x}$$

$$=\underset{x\to 0}{\mathrm{Lt}}\left[\log a+\frac{x}{2\,!}(\log a)^2+\ldots\ldots\right]=\log a.$$

Example 3. Evaluate :

$$\underset{x\to 0}{\text{Lt}} \frac{1-e^{-x}}{x}.$$

Sol. $$\underset{x\to 0}{\text{Lt}} \frac{1-e^{-x}}{x} = \underset{x\to 0}{\text{Lt}} \frac{1-\left(1-x+\frac{x^2}{2\,!}-\frac{x^3}{3\,!}+\ldots\ldots\right)}{x}$$

$$= \underset{x\to 0}{\text{Lt}} \left(1-\frac{x}{2\,!}+\frac{x^2}{3\,!}\ldots\ldots\right)=1.$$

Example 4. Evaluate :

$$\underset{x\to 0}{\text{Lt}} \frac{\log(1+x)}{x}.$$

Sol. $$\underset{x\to 0}{\text{Lt}} \frac{\log(1+x)}{x} = \underset{x\to 0}{\text{Lt}} \frac{x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\ldots}{x}$$

$$= \underset{x\to 0}{\text{Lt}} \left[1-\frac{x}{2}+\frac{x^2}{3}\ldots\ldots\right]=1.$$

Example 5. Evaluate :

$$\underset{x\to 0}{\text{Lt}} \frac{\tan x-\sin x}{x^3}.$$

Sol. $$\underset{x\to 0}{\text{Lt}} \frac{\tan x-\sin x}{x^3}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots\right)-\left(x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}\ldots\ldots\right)}{x^3}$$

$$= \underset{x\to 0}{\text{Lt}} \left(\frac{1}{3}+\frac{2}{15}x^2+\ldots\ldots+\frac{1}{6}-\frac{x^2}{120}+\ldots\ldots\right)=\frac{1}{2}.$$

REVISION EXERCISES

Exercise 1

Evaluate the following limits :

1. $\underset{x \to 0}{\text{Lt}} \dfrac{a^x-1}{e^x-1}$.

2. $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1+x)}{x}$.

3. $\underset{x \to \pi/2}{\text{Lt}} \dfrac{1-\sin^3 x}{\cos^2 x}$.

4. $\underset{x \to \infty}{\text{Lt}} (e^{-x} \cos x)$.

5. $\underset{n \to \infty}{\text{Lt}} \left[\dfrac{1^3+2^3+3^3+\ldots\ldots+n^3}{n^4}\right]$.

6. $\underset{x \to 3\pi/2}{\text{Lt}} \left(\dfrac{1+\sin x}{\cos x}\right)$.

7. $\underset{x \to \infty}{\text{Lt}} \dfrac{\sin x}{x}$.

8. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin 3x-\sin x}{x}$.

9. $\underset{x \to -\infty}{\text{Lt}} \dfrac{e^x}{x}$. ×

10. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin^3 x}{1-\cos x}$.

11. $\underset{x \to 1}{\text{Lt}} \dfrac{\log x}{1-x}$.

12. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x-e^a}{x-a}$.

13. $\underset{x \to \infty}{\text{Lt}} \left(\dfrac{x+1}{x}\right)^{x+2}$.

14. $\underset{x \to a}{\text{Lt}} \dfrac{x^m-a^m}{x^n-a^n}$.

15. $\underset{x \to \infty}{\text{Lt}} \sqrt{1+x^2}-x$.

16. $\underset{x \to 1}{\text{Lt}} \dfrac{\sqrt[3]{x-1}}{\sqrt{x-1}}$.

17. $\underset{x \to 0}{\text{Lt}} (\operatorname{cosec} x-\cot x)$.

18. $\underset{x \to 0}{\text{Lt}} \dfrac{\sinh x}{x}$.

19. $\underset{x \to 0}{\text{Lt}} \dfrac{\cosh x-1}{x^2}$.

20. $\underset{x \to 0}{\text{Lt}} \dfrac{\tanh x}{x}$.

Exercise 2

Differentiate the following w.r.t x from first principles :

1. $\sqrt{\sin x}$.
2. $\cos^3 x$.
3. $\sqrt{\log x}$.
4. $\tan(x^2)$.
5. $\log(\sqrt{x})$.
6. e^{x^2}.
7. $\sqrt{1+x^2}$.
8. $\dfrac{ax+b}{cx+d}$.
9. $\tan^{-1}(\sqrt{x})$.
10. $\cos^{-1}(x^2)$.
11. $\operatorname{cosec}^{-1}(ax)$.

12. $\cot^{-1}(ax+b)$. 13. $\log(ax+b)$. 14. $\cot(ax+b)$.
15. $x \sin x$. 16. $x^2 \tan x$. 17. $x^n \log x$.
18. $\cos x \cos 2x$. 19. $\sin 3x \sin x$.
20. $\dfrac{x}{\sin x}$.

Exercise 3

Differentiate the following w.r.t. x :

1. $x \log\left(\dfrac{x}{e}\right)$.
2. $x \tan^{-1} x - \log \sqrt{1+x^2}$.
3. $\log x(x+1)$.
4. $\cos^{-1}\left(\dfrac{a \cos x+b}{a+b \cos x}\right)$
5. $\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$.
6. $\tan^{-1}\left(\dfrac{a-x}{1+ax}\right)$
7. $\cot^{-1}\left(\dfrac{x-x^3}{1-x^4}\right)$.
8. $\left(1+\dfrac{1}{x}\right)^x+x^{1+\frac{1}{x}}$.
9. $\sin^{-1}(2x^2-1)$.
10. $\cos^{-1}(3x-4x^3)$.
11. $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)+2\sin^{-1} x$.
12. $x^{x^{x \ldots\ldots \text{to } \infty}}$.
13. $\sqrt{x+\sqrt{x+\ldots\ldots\infty}}$.
14. $\log \tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$
15. $\sin^m x \cos^n x$.
16. $\tan^{-1}\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)$.
17. $(\log x)^{\sin x}+(\sin x)^{\log x}$.
18. Find $\dfrac{dy}{dx}$ from :

$$9x^2+2hxy+by^2+2gx+2fy+c=0.$$

19. Find $\dfrac{dy}{dx}$: $x^m y^n=a^{m+n}$.
20. Find $\dfrac{dy}{dx}$ from :

$$\sqrt{1-x^2}+\sqrt{1-y^2}=a(x+y).$$

21. Find $\dfrac{dy}{dx}$ from :

$$x-\sqrt{1-y^2}=a[\sqrt{1-x^2}+y].$$

22. Differentiate :

$$(x+y)^{m+n}=x^m y^n.$$

23. Prove that the product of the values of $\frac{dy}{dx}$ from $x^2+y^2=a^2$ and $xy=k^2$ is unity.

Exercise 4

1. State and prove Leibnitz's theorem on nth derivative and hence find the nth derivative of :

(a) $x\, e^{ax} \cos bx$. (b) $x^2(ax+b)^n$.
(c) $x \log (ax+b)$. (d) $x^2\, a^{bx}$.
(e) $e^x \log x$.

2. Find the nth derivative of :

(a) $e^x \sin x$. (b) $\frac{2x+3}{3x+7}$.
(c) $e^x \cos^3 x$. (d) $e^x \sin x \cos x$.
(e) $\cos x \cos 2x \cos 3x$.
(f) $\cos^4 x$.
(g) $\frac{1}{x^3-6x^2+11x-6}$.
(h) $\frac{1}{x^2-5x+6}$.
(i) $e^{x \cos \alpha} \sin (x \sin \alpha)$.

3. If $y=\cos (m \sin^{-1} x)$, prove that

$$(1-x^2)y_2-xy_1+m^2y=0$$

and
$$(1-x^2)y_{n+2}+xy_{n+1}(2n+1)+(m^2+n^2)y_n=0.$$

4. If $y=(\sin^{-1} x)^2$, prove that

$$(1-x^2)y_2-xy_1=2$$

and
$$(1-x^2)y_{n+2}-xy_{n+1}(2n+1)-n^2y_n=0.$$

5. If $y=[x+\sqrt{1+x^2}]^n$, prove that

$$(1+x^2)y_2+xy_1-n^2y=0$$

and
$$(1+x^2)y_{n+2}+xy_{n+1}(2n+1)=0.$$

6. If $y=\sin (m\, \theta)$ and $x=\sin \theta$, show that

$$(1-x^2)y_2-xy_1+m^2y=0$$

and
$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(m^2-n^2)y_n=0.$$

7. $y=e^{ax} \sin bx$, show that

$$y_2-2ay_1+(a^2+b^2)y=0.$$

8. If $y=\sin (m \sin^{-1} x)$, show that

$$(1-x^2)y_2-xy_1+m^2y=0.$$

Exercise 5

Evaluate the following :

1. $\int \frac{x^2+2}{x^4+4}\, dx.$

2. $\int \frac{1}{1+\cos^2 x}\, dx.$

3. $\int \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}\, d\theta.$

4. $\int (e^x+a^x)^2\, dx.$

5. $\frac{a^x+e^x}{a^x e^x}\, dx.$

6. $\int \sqrt{1+\sin x}\, dx.$

7. $\int \frac{1}{x \log x \log\log x}\, dx.$

8. $\frac{1}{\cos x+\sin x}\, dx.$

9. $\int \frac{1-\tan x}{1+\tan x}\, dx.$

10. $\int \sqrt{\frac{1-\cos x}{1+\cos x}}\, dx.$

11. $\int \frac{2+x}{\sqrt{3+x}}\, dx.$

12. $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\, dx.$

13. $\int \frac{1}{x^3-7x^2+14x-8}\, dx.$

14. $\int \frac{x^2+x+4}{x^3+4x}\, dx.$

15. $\int \sin 2x \cos^3 x\, dx.$

16. $\int \sin x \sin 2x \sin 3x\, dx.$

17. $\int \frac{2x-3}{x^2-2x+2}\, dx.$

18. $\int \frac{1}{\sqrt{x-x^2}}\, dx.$

19. $\int \frac{1}{\sqrt{(1-x)(x-3)}}\, dx.$

20. $\int \frac{1}{(x+1)\sqrt{x^2+2x}}\, dx.$

21. $\int \frac{x^4+9}{x-1}\, dx.$

22. $\int \operatorname{cosec}^2 x \sec^2 x\, dx.$

23. $\int \frac{\sin 3x}{\sin x}\, dx.$

24. $\int 2x \cot (x^2)\, dx.$

25. $\int \frac{\log(\sqrt{x})}{x}\, dx.$

26. $\int \frac{\sin(e^x)}{e^{-x}}\, dx.$

27. $\int \frac{x}{\sqrt{1+x^2}}\, dx.$

28. $\int \frac{1+\sin x}{1+\cos x}\, dx.$

29. $\int (a^x+e^x+x^a)\, dx.$

30. $\int \sqrt{x-x^2}.dx.$

31. $\int \frac{x+1}{x^2-5x+6}\, dx.$

Exercise 6

1. Find the condition that the line $px+qy=1$ should touch $x^m y^n=a^{m+n}$.

2. Show that the tangent at any point of the curve $x^m y^n=a^{m+n}$

intercepted between the axes is divided in a constant ratio at the point of contact.

3. Show that the area of the triangle formed by the tangent at any point of the curve $xy=a^2$ and the axes is constant.

4. Find the length of normal drawn from (0, 0) upon the tangent at point 'θ' on the curve

$$x=a \sin^3 \theta \text{ and } y=a \cos^3 \theta.$$

5. Find the angle between

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

6. Find the tangent at (4, 8) to the curve $y^2=x^3$ and find where it cuts the curve again. Find tangent and normal to the curve at the other point.

7. Find the normal to $y^2=4ax$ making equal intercepts with the axes.

8. Show that the subrnormal to $y^2=4ax$ is constant.

9. Find the length of normal at any point 'θ' for the curve $x=a \cos \theta$, $y=b \sin \theta$.

10. Water in a conical cup is rising at the rate of 2″ per second. Find the rate of increase of its volume when h (height) $=4$ feet and α (semi-vertical angle) $=45°$.

11. The radius of a water ripple is increasing at the rate of 1″ per second. Find the rate of increase of its area when its radius $=2$ feet.

12. A tank is being filled up by water so that the water in the tank rises at the rate of 3″ per second. Find the rate of increase of its volume when the length and breadth of the tank are 3′ and 2′ respectively.

13. A particle moves subject to the law $x+A \cos (wt)$, where x and t are distance and time respectively. Show that the velocity becomes zero after $t=\frac{\pi}{w}$. Show that the acceleration is proportional to the distance.

14. A particle moves under a law $V=K\sqrt{a^2-x^2}$. Show that its acceleration is proportional to the distance covered.

KASHMIR UNIVERSITY PAPERS

1976

(For New Course Candidates)

Section A

1. (a) Define the limit of a function $f(x)$ when $x \to a$. Does $\underset{x\to 0}{\text{Lt}}\ e^{\frac{1}{x}}$ exists. Give reasons.

(b) Evaluate any *three* of the following :

(i) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin x}{x}$. (ii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{1-e^{-x}}{x}$.

(iii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{x}{\sqrt{4x^2+1}-1}$. (iv) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\log(1+x)}{x}$.

2. (a) Differentiate from first prineiples : $\sqrt{\tan x}$.

(b) Differentiate w.r.t. x any *two* :

(i) $\log\sqrt{\dfrac{1+\sin x}{1-\sin x}}$. (ii) $x^n e^x \log x$.

(iii) $(\tan x)^x + x^{\tan x}$.

3. (a) A triangle has two of its vertices at $(-a, 0)$ and $(a, 0)$ and the third (x, y) moving along the line $y=mx$. If A be its area, show that $\dfrac{dA}{dx}=ma$.

(b) If $y=x^{x^{x^{\cdots \text{ ad inf.}}}}$, find $\dfrac{dy}{dx}$.

4. (a) Prove that $\dfrac{x}{a}+\dfrac{y}{b}=1$ touches the curve $y=b\,e^{-\frac{x}{a}}$ where the curve crosses the y-axis.

(b) Find the equations of tangent and normal at any point of the curve $x=a(1+\sin t)$, $y=a(1\ \cos t)$.

Section B

5. (a) Find the nth derivative of $e^x \log x$.

(b) If $y=[\log\{x+\sqrt{1+x^2}\}]^2$, find the value of y_n at $x=0$.

6. (*a*) State the Maclaurin's Theorem and obtain the expansion of $f(x)=e^{a \sin^{-1} x}$.

(*b*) Integrate any *two* of the following :

(*i*) $\int \frac{dx}{x\sqrt{x^4-1}}$. (*ii*) $\int \frac{dx}{4-5\sin^2 x}$.

(*iii*) $\int e^x \frac{1+x\log x}{x}$.

7. Integrate any *three* of the following :

(*i*) $\int e^x(x\sin x+\cos x)\,dx$. (*ii*) $\int \sin^6 x\,dx$.

(*iii*) $\int \sin^{-1}\sqrt{\frac{a}{a+x}}\,dx$. (*iv*) $\int x\cot^{-1} x\,dx$.

8. Define the integral as the limit of a sum. Find the limits as n increases indefinitely of any *two* of the following :

(*i*) $\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots\ldots+\frac{n}{n^2+n^2}$.

(*ii*) $\frac{1}{n^2}+\frac{2}{n^2}+\ldots\ldots+\frac{n-1}{n^2}$.

(*iii*) $\frac{1}{n}+\frac{1}{\sqrt{(n^2-1^2)}}+\frac{1}{\sqrt{(n^2-2^2)}}+\ldots\ldots+\frac{1}{\sqrt{\{n^2-(n-1)^2\}}}$.

(*For Old Course Candidates*)

4. (*a*) Illustrate the idea of a limit by examples.

(*b*) Evaluate any *three* of the following :

(*i*) $\underset{x\to 3}{\text{Lt}}\ \frac{x^2-9}{x-3}$. (*ii*) $\underset{x\to 0}{\text{Lt}}\ \frac{\sqrt{1+x}-1}{x}$.

(*iii*) $\underset{x\to 0}{\text{Lt}}\ \frac{a^x-1}{x}$. (*iv*) $\underset{\theta\to\pi/2}{\text{Lt}}\ \frac{\cot\theta-\cos\theta}{\cos^3\theta}$.

5. (*a*) Differentiate *ab initio* $\sin\sqrt{x}$ or $\cos^2 x$.

(*b*) Find $\frac{dy}{dx}$ of any *three* :

(*i*) $y=\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$.

(*ii*) $x=\frac{3at}{1+t^3}$, $y=\frac{3at^2}{1+t^3}$.

(*iii*) $y=\sqrt{x+\sqrt{x+\sqrt{x+\ldots\ldots \text{to } \infty}}}$.

(*iv*) $y=\sqrt{\frac{1+x}{1-x}}$.

6. (*a*) Differentiate (any *three*) :

(*i*) $\tan^{-1}\dfrac{2x}{1-x^2}$ w.r.t. $\sin^{-1}\dfrac{2x}{1+x^2}$.

(*ii*) $(\tan x)^x+x^{\tan x}$ w.r.t. x. (*iii*) x^{x^x} w.r.t. x.

(*iv*) $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$ w.r.t. x.

(*b*) If $\sin y=x\sin(a+y)$, prove that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$.

Or

(*b*) Prove that $\dfrac{x}{a}+\dfrac{y}{b}=2$ touches the curve $\left(\dfrac{x}{a}\right)^n\left(\dfrac{y}{b}\right)^n=2$ at (a, b) for all values of n.

7. (*a*) If a point is moving in a straight line and distance in feet from a point in the line after t seconds is given by $x=5+2t+4t^3$, find the acceleration at the end of $3\frac{1}{2}$ seconds.

(*b*) Find the equation of the normals to the curve $(y-3)(y-6)=x^2$ at the point where $y=7$.

8. (*a*) Find the *n*th derivative of $\cos^3 x$ or $\dfrac{x+2}{3x+7}$.

(*b*) If $y=[x+\sqrt{1+x^2}]^m$, show that :

(*i*) $y_2(1+x^2)+xy_1-m^2y=0$.

(*ii*) $y_{n+2}(1+x^2)+(2x+1)xy_{n+1}+(n^2+m^2)y_n=0$.

9. Integrate (any *three*) :

(*i*) $\dfrac{x^2+2x+3}{x+1}$. (*ii*) $\sqrt{1+\sin x}$.

(*iii*) $\sqrt{\dfrac{a+x}{a-x}}$. (*iv*) $\sin^{-1}x$. (*v*) x^3e^x.

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Section C

5. (*a*) Explain with regard to a function $f(x)$ the statement $x=a$ and $x\to a$ giving a few examples.

(*b*) Evaluate any *three* of the following :

(*i*) $\underset{x\to a}{\text{Lt}}\dfrac{x^n-a^n}{x-a}$. (*ii*) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{\sin px}{\sin qx}\right)^x$.

(*iii*) $\underset{x\to\infty}{\text{Lt}}\left(1+\dfrac{1}{x}\right)^x$. (*iv*) $\underset{x\to 0}{\text{Lt}}\dfrac{e^x-1}{x}$.

6. (*a*) Define differential co-efficient and differentiate from first principle $\frac{1}{\sqrt{x+c}}$.

(*b*) Differentiate w.r.t. x any *two* of the following :

(*i*) $\left(1+\frac{1}{x}\right)^x+(x)^{1+\frac{1}{x}}$.

(*ii*) $\tan^{-1}\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)$. (*iii*) $\cos(x^x)$.

(*iv*) $\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}$.

7. (*a*) The diameter of an expanding smoke ring at time t is proportional to t^2. If the diameter is 6 cm after 6 seconds, at what rate is it then changing ?

(*b*) If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, prove that

$$\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.$$

8. (*a*) Give the geometrical interpretation of the derivative of a function and use it to obtain the equations of tangent and normal to the curve $y=f(x)$ at a point (x, y) on it.

(*b*) Find the equations of the tangent and the normal at the point "t" to the curve whose equations are

$$x=a\cos^3 t,\ y=a\sin^3 t.$$

Section D

9. (*a*) Find the nth derivative of $e^x\cos^3 x$.

(*b*) If $y^{\frac{1}{m}}+y^{-\frac{1}{m}}=2x$, prove that

$$(x^2-1)y_{n+2}+(2x+1)xy_{n+1}+(n^2-m^2)y_n=0.$$

10. (*a*) Apply Maclaurin's Theorem to expand $\tan x$ as far as the term in x^5.

(*b*) Integrate any *two* of the following :

(*i*) $\int\frac{dx}{1+\sin x}$. (*ii*) $\int\frac{x^5\,dx}{\sqrt{1+x^2}}$.

(*iii*) $\int\frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$. (*iv*) $\int\cos^5 x\,dx$.

11. Evaluate any *three* :

(*i*) $\int \frac{xe^x\,dx}{(x+1)^2}$. (*ii*) $\int x \tan^{-1} x\,dx$.

(*iii*) $\int \frac{x \sin^{-1} x\,dx}{\sqrt{1-x^2}}$. (*iv*) $\int e^{ax} \cos(bx+c)\,dx$.

(*v*) $\int \operatorname{cosec}^3 x\,dx$.

12. (*a*) Define $\int_a^b f(x)$ as the limit of a sum and evaluate $\int_a^b x^2\,dx$ as the limit of the sum.

(*b*) If $y=\sin(\sin x)$, prove that $y_2+y_1 \tan x+y \cos^2 x=0$.

ANSWERS

Exercise II

2. $3, -6, -5, \dfrac{1+2x-5x^2}{x^2}, a^2+2ah+h^2+2a+2h-5.$

3. $1, \dfrac{\sqrt{3}}{2}, 0.$ **7.** $2a+h.$

15. (*i*) $x=1,$ (*ii*) $x=2, 3.$ (*iii*) $k\pi.$

16. $\pi, 2\pi, 2\pi, \pi.$

Exercise III

1. 10. **2** $\frac{2}{3}$. **3.** ma^{m-1}. **4.** $\frac{1}{2}$. **5.** $\frac{3}{5}$.

6 $\frac{3}{2}$. **7.** 1. **8.** 1. **9.** 2. **10.** 0.

11. $\frac{1}{2}$. **12.** $\dfrac{a}{1-\gamma}$. **13.** $\frac{1}{4}$. **14.** $\frac{1}{2}$. **15.** $\dfrac{m}{n}$.

16. $\sec^2 x$. **17.** $\frac{1}{2}$. **18.** $\frac{3}{2}$. **19.** $\frac{1}{2}$. **20.** $\frac{1}{2}$.

26. $\dfrac{m}{n}$. **27.** 0. **28.** 1. **29.** $-\dfrac{1}{25}$. **30.** 1.

Exercise IV

1. (*i*) $5x^4.$ (*ii*) $-\dfrac{1}{x^2}.$ (*iii*) $\dfrac{1}{2\sqrt{x}}.$

(*iv*) $3x^2.$ (*v*) $-\dfrac{2}{x^3}.$ (*vi*) $20(2x+3)^9.$

(*vii*) $-\frac{26}{3}(8-13x)^{-\frac{1}{3}}.$ (*viii*) $-\frac{5}{2}(5x+7)^{-\frac{3}{2}}.$

2. (*i*) $1-2x.$ (*ii*) $3x^2-12x+11.$

(*iii*) $4x^3+15x^2-14x+4.$ (*iv*) $-\dfrac{2}{x^3}.$ (*v*) $2x+3-\dfrac{8}{x^3}.$

(*vi*) $m.$ (*vii*) $-\dfrac{28}{x^5}-\dfrac{56}{x^8}.$

3. (*i*) $mpx^{m-1}+nqx^{n-1}.$ (*ii*) $4x^3+12x^2+12x+4.$

(*iii*) $-2+6t-12t^2.$ (*iv*) 27. (*v*) 11.

4. (*i*) $15x^2.$ (*ii*) $6\,(3x+4).$ (*iii*) $-\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{x^3}}.$

(iv) $2ax+b$. (v) $-\dfrac{3}{(3x+1)^2}$. (iv) $\dfrac{1}{3\,.\,\sqrt[3]{x^2}}$.

(vii) $\dfrac{16}{(5x+7)^2}$. (viii) $-\dfrac{1}{2\sqrt{ax+b}}$. (ix) $\dfrac{1}{2\sqrt{x}}\left(1-\dfrac{1}{x}\right)$.

(x) $-\dfrac{a}{2\,.\,\sqrt{(ax+b)^3}}$. (xi) $\dfrac{ad-bc}{(cx+d)^2}$. (xii) $2x+\dfrac{18}{(x+3)^2}$.

(xiii) $-\dfrac{1}{2}\;\dfrac{1}{(x+c)^{3/2}}$.

Exercise V

1. $8(3x^2+7x+5)^4(6x+7)$.
2. $n(ax^2+bx+c)^{n-1}(2ax+b)$.
3. $\dfrac{-3x^5}{\sqrt{2-x^6}}$.
4. $\dfrac{-6(7x^6+12x^2+21)}{(x^7+4x^3+21x)^7}$.
5. $\dfrac{-4(16x-7)}{(8x^2-7x+9)^{3/2}}$.
6. $\dfrac{m}{n}(3px^2-2qx+r)(px^3-qx^2+rx+s)^{\frac{m-n}{n}}$.
7. $\dfrac{2px}{q}\;(2x^2+3)(x^4+3x^2-1)^{\frac{p-q}{q}}$.
8. $\dfrac{-x}{\sqrt{a^2-x^2}}-\dfrac{x}{(a^2+x^2)^{3/2}}$.
9. $-\dfrac{1}{5}\cdot\dfrac{14x}{\sqrt[5]{(7x^2-3)^4}}$.
10. $20x(c^2+x^2)[c^2+(c^2+x^2)^2]^4$.
11. $30x^4-27x^2+8x$.
12. $\left(1-\dfrac{1}{x^2}\right)\left(2x-\dfrac{3}{x}\right)+\left(x+\dfrac{1}{x}\right)\left(2+\dfrac{3}{x^2}\right)$.
13. $5x^4+21x^2+20x$.
14. $(p+m)x^{m+p}-1+pnx^{p-1}+mpx^{m-1}$.
15. $2(6x^2-x-5)$.
16. $\left(1-\dfrac{1}{x^2}\right)\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)+\dfrac{1}{2}\left(x+\dfrac{1}{x}\right)\left(\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{x^3}}\right)$.
17. $\dfrac{8}{3}\;x^{5/3}+\dfrac{2}{\sqrt[3]{x}}$.
18. $-6x(3x^2+4)(1-x^2)(5x^2+2)$.
19. $\dfrac{x(3bx^5+5bx^3a-2c)}{(x^2+a)^2}$.
20. $\dfrac{1}{(1+x)^{1/2}(1-x)^{3/2}}$.
21. $\dfrac{1}{2}\cdot\dfrac{b-2ac}{(x-2a)^{1/2}(b-cx)^{3/2}}$.
22. $\dfrac{-2(x^2+2x+5)}{(x^2+2x-3)^2}$.
23. $\dfrac{a[a-\sqrt{a^2-x^2}]}{x^2\sqrt{a^2-x^2}}$.
24. $\dfrac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}$.

Exercise VI

1. $\frac{2t}{t^2-1}$.
2. $-\frac{b}{a}\cdot\frac{1-t^2}{2t}$.
3. $\frac{1}{t}$.
4. $-\frac{1}{t^2}$.
5. $\frac{t(2-t^3)}{1-2t^3}$.
6. $\frac{x+g}{y+f}$.
7. $-\frac{y^{2/3}}{x^{2/3}}$.
8. $-\frac{y^{1/3}}{x^{1/3}}$.
9. $\frac{x^2-y}{y^2-x}$.
10. $-\frac{ax+hy}{hx+by}$.
11. $-\frac{x^{n-1}}{y^{n-1}}$.
12. $-\frac{y}{x}$.
13. $-\frac{ax+hy+g}{hx+by+f}$.
14. $\frac{n}{3}x^{n-3}$.
15. $\frac{35x^4-22x}{14x-15}$.
16. $\frac{1-x^2}{3x^2(1+x^2)^2}$.
17. $-\frac{x+\sqrt{1-x^2}}{x}$.
18. $\frac{ad-bc}{a'd'+b'c'}\cdot\frac{(c'x+d')^2}{(ax+b)^2}$.
21. $\frac{1+nx^{n+1}-(n+1)x}{(1-x)^2}$.

Exercise VII

I. 1. $-m\sin mx$.
2. $\frac{\pi}{180}\cos x^0$.
3. $-m\cos^{m-1}x\sin x$.
4. $9\tan^2(3x+6)\sec^2(3x+6)$.
5. $8a(ax+b)\sin^3(ax+b)^2\cos(ax+b)^2$.
6. $\cos x\sec^2(\sin x)$.
7. $2x\sec^2 x\tan x$.
8. $m\sin^{m-1}x\cos^{n+1}x-n\sin^{m+1}x\cos^{n-1}x$.
9. $\frac{-2\cos x}{(1+\sin x)^2}$.
10. $\frac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$.

II. 1. $-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$.
2. $\frac{1}{2}$.
3. 1.
4. $3\sec^2 3x$.
5. $\frac{1}{2}$.
6. $\frac{3}{2}$.
7. $-\frac{1}{2}$.
8. $-\frac{1}{2}$.

III. 1. $\frac{2}{1+x^2}$.
2. $\frac{2}{1+x^2}$.
3. $\frac{3}{1+x^2}$.
4. $\frac{3}{\sqrt{1-x^2}}$.
5. $\frac{1}{2(1+x^2)}$.
6. $-\frac{1}{1+x^2}$.
7. $-\frac{2x}{1+x^4}$.

IV. 1. $\sec^3 x$.
2. $2x^2\cos x^4$
3. $-\cot x$.
4. $\frac{x^2\cos x^3}{\sin^2 x\cos x}$.
5. 1.
6. $\frac{1}{2}$.
7. $\frac{1}{2}$
8. $\frac{2}{x}$.

V. 1. $\frac{1}{2}\,\dfrac{\cos x}{\sqrt{\sin x}}$.

2. $\dfrac{\sqrt{\cos x}}{4\sqrt{x}\sqrt{\sin\sqrt{x}}}$.

3. $\dfrac{\sec^2\sqrt{x}}{4\sqrt{x\tan\sqrt{x}}}$.

4. $\dfrac{x\sec^2\sqrt{1+x^2}}{2\sqrt{(1+x^2)\tan\sqrt{1+x^2}}}$.

VIII. 1. $-\dfrac{b}{a}\cot\theta$.

2. $-\dfrac{b}{a}\tan\theta$.

3. $\dfrac{2b}{3a}\cos\theta$.

4. $\tan\dfrac{\theta}{2}$.

IX. 1. $-3\sin 3x$.

2. $5\sec 5x\tan 5x$.

3. $\dfrac{1}{2\sqrt{x}}\cos\sqrt{x}$.

4. $-2x\cos x\sin x$.

5. $-\dfrac{\sin x}{2\sqrt{\cos x}}$.

6. $2x\sec^2 x^2$.

7. $\sin x+x\cos x$.

8. $\dfrac{\sec^2 x}{2\sqrt{\tan x}}$.

Exercise VIII

I. 1. $\dfrac{3}{\sqrt{1-9x^2}}$.

2. $-\dfrac{3}{\sqrt{1-(3x+2)^2}}$.

3. $\dfrac{1}{x^2+a^2}$.

4. $\dfrac{2}{x\sqrt{x^4-1}}$.

5. 1.

6. $\dfrac{\sec^2(\tan^{-1}x)}{1+x^2}$.

7. $-\dfrac{1}{2x\sqrt{x^2-1}}$.

8. 0.

9. $\sec^2 x\tan^{-1}x+\dfrac{\tan x}{1+x^2}$.

10. $\dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$.

II. 1. $-\dfrac{4x^2+12x+10}{2x\sqrt{x^2-1}}$.

2. $\dfrac{1}{\sqrt{1+x^4}}$.

3. 1.

IV. 1. $-\dfrac{x}{\sqrt{1-x^2}}$.

2. $-\dfrac{1}{2\sqrt{1-x^2}}$.

3. $\frac{1}{2}\cdot\dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$.

4. $\dfrac{\cos^{-1}x-x\sqrt{1-x^2}}{(1-x^2)^{3/2}}$.

Exercise IX

I. 1. $\dfrac{2x}{x^4-1}$. 2. $\dfrac{2}{\sqrt{1+x^2}}$. 3. $\sec x$.

4. $\sec x$. 5. $2 \operatorname{cosec} x$. 6. $\dfrac{1}{x}+\sec x \operatorname{cosec} x$.

7. $\dfrac{1}{(1-x^2)\tan^{-1} x}$. 8. $\dfrac{1}{x \log x}$.

9. $\dfrac{x^2-1}{x^2-4}$. 10. $\operatorname{cosec} x$.

II. 1. $(x+1)\, e^x$. 2. $ae^{ax}+2e^{2x}$.

3. $e^x (\tan x+\sec^2 x)$. 4. $-(e^x-e^{-x}) \sin (e^x+e^{-x})$.

5. $e^x \cos e^x$.

6. $2 \sec^2 (2x+3)a^{\tan (2x+3)} \log a$.

7. $\dfrac{a^{\sin^{-1} x}}{\sqrt{1-x^2}} \log a$.

8. $\left(\tan^{-1} x+\dfrac{x}{1+x^2}\right) e^{x \tan^{-1} x}$.

9. $\dfrac{x}{\sqrt{1+x^2}}\, e^{\sqrt{1+x^2}}$. 13. $\dfrac{3x^2+e^x}{x^3+e^x}$.

IV. 1. $\dfrac{1}{x[1+(\log x)^2]}$. 2. $\dfrac{\cos (\log \tan x)}{\sin x \cos x}$.

3. $\dfrac{\cos\left(\log \sqrt{\dfrac{x}{x+1}}\right)}{x(x+1)}$.

4. $\dfrac{1-x^2}{1+x^2+x^4}$. 5. $\dfrac{-12}{(x^2-3)^2}\times (a)^{\frac{x^3+3}{x^2-3}}\times(\log a)$.

6. $\dfrac{2ab}{a^2 \cos^2 x-b^2 \sin^2 x}$. 7. $\dfrac{1}{x \log x \log (\log x)}$.

Exercise X (Section A)

1. $\sin x\times\log x\times e^x\times \sqrt{x}\times\left\{ \cot x+\dfrac{1}{x \log x}+\dfrac{2x+1}{2x}\right\}$

2. $\frac{1}{2} \sqrt{x} \sqrt{\sin x} \sqrt{\log x}\left\{\dfrac{1}{x}+\cot x+\dfrac{1}{x \log x}\right\}$.

3. $x^x (1+x+x \log x)$.

4. $(\tan x)^{\cot x} . \operatorname{cosec}^2 x (1-\log \tan x)$.

5. $(\sin^{-1} x)^x \left\{ \log \sin^{-1} x + \dfrac{x}{\sin^{-1} x \sqrt{1+x^2}} \right\}$.

6. $x^{\sin x} \left\{ \dfrac{1}{x} \sin x + \cos x \log x \right\}$.

7. $(\sin x)^{\log x} \left\{ \dfrac{1}{x} \log \sin x + \cot x \log x \right\}$.

8. $-x^x \log ex \,.\, \sin x^x$.

9. $b^x a^{b^x} \log a \,.\, \log b$.

10. $x^n x^{x^n} \left\{ \dfrac{1}{x} + \log x + (\log x)^2 \right\}$.

11. $x^x (1+\log x) + \left(\dfrac{1}{x^2} \right) x^{1/x} (1-\log x)$.

12. $(\sin x)^{\cos x} (\cot x \cos x - \sin x \log x) + (\cos x)^{\sin x}$
$\times (-\tan x \sin x + \cos x \log \cos x)$.

13. $(\sec x)^{\operatorname{cosec} x} (\sec x - \operatorname{cosec} x \cot x \log \sec x)$
$+ (\operatorname{cosec} x)^{\sec x} (\sec x \tan x \log \operatorname{cosec} x - \operatorname{cosec} x)$.

14. $(\tan x)^x (\log \tan x + 2x \operatorname{cosec} 2x)$
$+ x^{\tan x} \left(\sec^2 x \log x + \dfrac{1}{x} \tan x \right)$.

15. $\dfrac{2^x \cot x}{\sqrt{x}} \left[\log 2 - 2 \operatorname{cosec}^2 x - \dfrac{1}{2x} \right]$.

16. $\dfrac{x \cos^{-1} x}{\sqrt{1-x^2}} \left[\dfrac{1}{x} - \dfrac{1}{\sqrt{1-x^2} \cos^{-1} x} + \dfrac{x}{1-x^2} \right]$.

Section (B)

1. $\cosh^3 x$, $\dfrac{-2(1+\cos x \cosh x)}{(\sinh x + \sin x)^2}$.

2. $\tanh x$, $\dfrac{1}{\sinh x \cosh x}$ $a^{\sinh x} \,.\, \cosh x \,.\, \log a$.

3. $\dfrac{1}{\sqrt{x^2-1}}$, $\sec x$.

4. $\operatorname{sech} x$ for each part. 5. $\sec x$ for each part.

Exercise XI

1. $3024x^5$. 2. $-\dfrac{6}{x^4}$.

3. $-\dfrac{120a^5}{(ax+b)^6}$. 4. $-\dfrac{3}{(3-x)^2}$.

5. $4\tan^2 x\sec^2 x+2\sec^4 x-4\cot^2 x\,\mathrm{cosec}^2 x-2\,\mathrm{cosec}^4 x.$

6. $\left\{\dfrac{n^2}{x^2-1}-\dfrac{nx}{(x^2-1)^{3/2}}\right\}\left\{x+\sqrt{x^2-1}\right\}^n.$

14. $-\frac{3}{2}.$ **15.** $-\dfrac{b}{a^2}\,\mathrm{cosec}^3\,\theta.$

Exercise XII

1. $\dfrac{(-1)^n\,\underline{|n}}{(1+x)^{n+1}}.$ **2.** $\dfrac{n\,!}{(a-x)^{n+1}}.$

3. $\dfrac{b^n\,.\,n\,!}{(a-bx)^{n+1}}.$ **4.** $\dfrac{(-1)^{n-1}\,(n\,!)\,3^{n-1}}{(3x+7)^{n+1}}.$

5. $\dfrac{(-1)^n\,\underline{|n}+1\,.\,2^n}{(2n+3)^{n+2}}.$

6. $\frac{1}{2}\cos(x+\frac{1}{2}n\pi)-\frac{1}{2}\,.\,3^n\,.\,(3x+\frac{1}{2}n\pi).$

7. $(b-a)\,.\,2^{n-1}\times\cos\,(2x+\frac{1}{2}n\pi).$

8. $2^n\,e^{2x}+5^n\,a^{5x}\,(\log 5)^n.$

9. $\frac{1}{4}\{2^n\cos(2x+\frac{1}{2}n\pi)+4^n\cos(4x+\frac{1}{2}n\pi)+6^n\cos(6x+\frac{1}{2}n\pi)\}.$

10. $-2^{n-1}\cos(2n+\frac{1}{2}n\pi).$

11. $\frac{1}{8}\{4^n\cos(4x+\frac{1}{2}n\pi)+2^{n+2}\cos(2x+\frac{1}{2}n\pi)\}.$

12. $\frac{1}{2}e^x\{2^{n/2}\cos(x+\frac{1}{2}n\pi)+10^{n/2}\cos(3x+n\tan^{-1}3)\}.$

13. $e^x\,.\,2^{n/2}\left[\cos\left(x+\dfrac{n\pi}{4}\right)+\sin\left(x+\dfrac{n\pi}{4}\right)\right].$

14. $e^{x\cos\alpha}\,.\,\cos(x\sin\alpha+n\alpha).$

15. $\frac{3}{4}\sin\left(x+n\,\dfrac{\pi}{2}\right)-\dfrac{3^n}{4}\,\sin\left(3x+\dfrac{n\pi}{2}\right).$

16. $\frac{1}{2}e^x+\frac{1}{2}\gamma^n e^x\cos(2x+n\theta)$ where $\gamma=\sqrt{5}$ and $\theta=\tan^{-1}2.$

17. $\frac{3}{4}\gamma^n\,e^x\sin(x+n\theta)-\frac{1}{4}R^n\,e^x\sin(3x+n\varphi)$

where $\gamma=\sqrt{2},\ \theta=\dfrac{\pi}{4}$

$R=10,\ \varphi=\tan^{-1}3.$

18. $2^n\,e^x\sin\left(\sqrt{3}x+\dfrac{n\pi}{3}\right).$

19. $\frac{1}{2}e^{5x}\left[(50)^{n/2}\sin\left(5x+\dfrac{n\pi}{4}\right)+(26)^{n/2}\sin\left(x+n\tan^{-1}\dfrac{1}{5}\right)\right].$

20. $5^n\,e^{3x}\sin(4x+n\tan^{-1}\frac{4}{3}).$

Exercise XIII

1. $x\,[x^2-3n(n-1)]\cos\left(x+\dfrac{n\pi}{2}\right)$

$$+n[3x^2-(n-1)(n-2)]\sin\left(x+\frac{n\pi}{2}\right).$$

2. $2(-1)^{n-1}\,.\,(n-3)\,!\,x^{-n+2}$.

3. $e^x\,[\log x+{}^nC_1\,x^{-1}-{}^nC_2\,x^{-2}+\ldots\ldots+(-1)^{n-1}\,.\,(n-1)\,!\,x^{-n}]$.

4. $a^x\,[x^2\,(\log a)^n+2nx\,(\log a)^{n-1}+n(n-1)(\log a)^{n-2}]$.

5. $e^{ax}\,[x^2a^n+2nx\,a^{n-1}+n(n-1)a^{n-2}]$.

6. $e^x\left[\,x^n+\dfrac{n^2}{1\,!}\,x^{n-1}+\dfrac{n^2\,(n-1)^2}{2\,!}\,x^{n-2}+\ldots\ldots\right]$

7. $(a^2+b^2)^{n/2}\,.\,xe^{ax}\sin\left\{\left(bx+n\tan^{-1}\dfrac{b}{a}\right)\right\}+n(a^2+b^2)^{(n-1)/2}$

$$\times e^{ax}\sin\left\{bx+(n-1)\tan^{-1}\frac{b}{a}\right\}.$$

11. When n is odd, $(y_n)_0$ is

$$\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\ldots\{(3^2+a^2)(1+a^2)\,.\,a\}.$$

When n is even, $(y_n)_0$ is

$$\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\ldots\ldots(4^2+a^2)(2^2+a^2)\,a^2.$$

14. When n is odd,

$$(y_n)_0=\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots\ldots(m^2-3^2)\times(m^2-1)^m.$$

When n is even,

$$(y_n)_0=\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots\ldots(m^2-2^2)m^2.$$

Exercise XIV

1. (*i*) $2\pi\gamma$. (*ii*) $2\pi(\gamma+h)$. (*iii*) $4\pi\gamma^2$.

2. $-ake^{-kt}\sin(\omega t+a)+a\omega e^{-kt}\cos(\omega t+a)$.

3. $-\frac{3}{4}$, $\frac{4}{3}$.

4. 72π square inches per second.

5. (*i*) $92\frac{3}{4}$ ft./sec. (*ii*) 84 ft./sec^2. (*iii*) 150 ft.

11. $\frac{1}{8}t$.

Exercise XV

1. (*i*) $yY=2a(X+x)$; $2a(Y-y)+y(X-x)=0$.

(*ii*) $a^2Y-3x^2X+2x^3=0$;

$3x^2(Y-y)+a^2(X-x)=0$.

(*iii*) $(x^2-ay)X+(y^2-ax)Y=axy$;

$(Y-y)(x^2-ay)-(X-x)(y^2-ax)=0$.

(*iv*) $ty=x+at^2$; $y+tx=2at+at^3$.

(*v*) $bx-ay\sin\theta=ab\cos\theta$;

$ax\sin\theta+by=(a^2+b^2)\tan\theta$.

(*vi*) $x \sin \frac{1}{2}t - y \cos \frac{1}{2}t = at \sin \frac{1}{2}t$;
$x \cos \frac{1}{2}t + y \sin \frac{1}{2}t = at \cos \frac{1}{2}t + 2a \sin \frac{1}{2}t$.

2. (*i*) $\dfrac{xx'}{a^2} + \dfrac{yy'}{b^2} = 1$; $\dfrac{y'}{b^2}(x - x') = \dfrac{x'}{a^2}(y - y')$.

Tangent at $(a \cos \theta, b \sin \theta)$ is

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1.$$

Normal at $(a \cos \theta, b \sin \theta)$ is

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2.$$

(*ii*) Tangent : $x'y + y'x = 2c^2$
Normal : $xx' - yy' = x'^2 - y'^2$.

(*iii*) Tangent : $3x + 2y = 3$
Normal : $2x + 3y - 15 = 0$.

(*iv*) Tangent : $y - x + a = 0$
Normal : $x + y = 3a$.

(*v*) Tangent : $x = 2a$
Normal : $y = 0$.

4. $(2, 16), (2, -16)$.
7. $y = 2x + 3$.
8. $x + 3y = \pm 9$.
11. $a^2 l^2 + b^2 m^2 = n^2$.
13. $(a \cos \alpha)^{\frac{m}{m-1}} + (b \sin \alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$.

Exercise XVI

1. $\dfrac{\pi}{3}$.
2. $\dfrac{\pi}{4}$.
3. $\tan^{-1} 3$.
4. $\dfrac{\pi}{2}$; $\tan^{-1}\left(\dfrac{3}{4}\right)$.
5. $\tan^{-1}\left(\dfrac{3}{2} \quad \dfrac{a^{1/3}\, b^{1/3}}{a^{2/3} + b^{2/3}}\right)$.
6. $\tan^{-1}(\sqrt[3]{16})$.
8. $\tan^{-1}(\frac{1}{4}\frac{3}{1})$.
9. $90°$; $\tan^{-1}\left(\dfrac{3}{2^{2/3} + 2^{4/3}}\right)$.

Exercise XVII

6 $a \sin^2 \theta$, $a \tan \theta \sin^2 \theta$, $a \sin^2 \theta \cos \theta$, $a \sin^2 \theta \tan \theta$.

Exercise XVIII

1. $\dfrac{x^9}{8}$; x^3 ; $2x^{1/2}$; $\dfrac{(x+8)^6}{6}$; $\dfrac{(2x+3)^6}{12}$.

2. (*i*) $-\dfrac{1}{12(3x+4)^4}$.
(*ii*) $\dfrac{x^4}{4} - \dfrac{x^5}{5} + \frac{1}{3}x^6$.

(iii) $3x+2x^2+5\log x$.

(iv) $-\dfrac{1}{x}-\dfrac{1}{2x^2}+\dfrac{1}{3x^3}+\dfrac{3}{4x^4}$.

(v) $-\dfrac{a}{2cx^2}-\dfrac{b}{cx}+\log x$.

(vi) $\log(x-5)$.

(vii) $\dfrac{1}{b}\log(a-bx)$.

(viii) $\theta-\theta^2+\theta^3-\dfrac{\theta^4}{2}+\dfrac{\theta^5}{5}$.

(ix) $\dfrac{x^4}{4}-\dfrac{1}{2x^2}+\frac{3}{2}x^2+3\log x$.

(x) $\frac{2}{3}(x+1)^{3/2}+2(x+1)^{1/2}$.

(xi) $\dfrac{2}{3(a-b)}\{(x+a)^{3/2}-(x+b)^{3/2}\}$.

(xii) $-\dfrac{1}{3a}\dfrac{1}{(ax+b)^3}$.

3. (i) $x+\log(x-4)^9$.

(ii) $4x+\log(x-4)^{17}$.

(iii) $x-\frac{4}{3}\log(3x+4)$.

(iv) $\dfrac{x^3}{3}-\frac{3}{2}x^2+9x-34\log(x+3)$.

(v) $\frac{3}{4}x^2-\frac{3}{4}x+\frac{7}{8}\log(2x+1)$.

(vi) $\dfrac{x^2}{2}+x+2\log(x+1)$.

(vii) $\dfrac{x^3}{3}+\frac{3}{2}x^2+9x$.

(viii) $\frac{2}{3}(x+3)^{3/2}-6(x+3)^{1/2}$.

(ix) $\frac{2}{3}(x+a)^{3/2}+2(b-a)(x+a)^{1/2}$.

(x) $\frac{2}{5}(x+3)^{5/2}-2(x+3)^{3/2}$.

4. (i) $3\sin\dfrac{x}{3}$.

(ii) $\cos x$.

(iii) $\cos x$.

(iv) $\tan x$.

(v) $\dfrac{\tan 4x}{4}$.

(iv) $\dfrac{\tan ax}{a}$.

(vii) $\tan x-x$.

(viii) $\frac{1}{12}(\sin 3x+9\sin x)$.

(ix) $x+\dfrac{\cos 2x}{2}$.

(x) $\tan x-\cot x$.

(xi) $\frac{1}{2}\tan\theta$.

(xii) $\tan x+\cot x$.

(xiii) $-\frac{1}{2}\left(\dfrac{\cos 5x}{5}+\cos x\right)$.

(xiv) $\dfrac{1}{2}\left(\dfrac{\sin 5x}{5}+\sin x\right)$.

(*xv*) $\frac{1}{2}\left[-\frac{\cos(p+q)x}{p+q}+\frac{\cos(p-q)x}{p-q}\right]$.

(*xvi*) $\operatorname{cosec} x-\cot x$. (*xvii*) $-(\cot x+\operatorname{cosee} x)$.

(*xviii*) $x-\tan x+\sec x$. (*xix*) $2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)$.

(*xx*) $2\sqrt{2}\sin\frac{x}{2}$.

5. (*i*) $-e^{-x}$. (*ii*) $\frac{e^{-ax}}{-a}$. (*iii*) $\frac{10^x}{\log 10}$.

(*iv*) e^x+e^{-x}. (*v*) $\frac{e^{2x}}{2}-\frac{e^{-2x}}{2}+2x$.

(*vi*) $\frac{x^{n+1}}{n+1}+\frac{n^x}{\log n}+x$.

6. (*i*) $\frac{1}{4}$. (*i*) $\frac{1}{5}$. (*iii*) $1-\frac{\pi}{4}$.

(*iv*) $\frac{7}{8}\log 3$. (*v*) $\frac{1}{4}\log 5$. (*vi*) $\frac{\pi}{2}$.

(*vii*) $\sqrt{3}$. (*viii*) $-\frac{2}{\sqrt{3}}$.

Exercise XIX

1, (*i*) $\frac{1}{2}\tan^{-1}\frac{x}{2}$. (*ii*) $\frac{1}{4}\tan^{-1}\frac{y}{4}$.

(*iii*) $\frac{1}{2}\tan^{-1}(2x)$. (*iv*) $\frac{1}{4a}\tan^{-1}\frac{x}{4a}$.

(*v*) $\tan^{-1}(x+2)$. (*vi*) $\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x}{2\sqrt{2}}$.

2. (*i*) $\sin^{-1}x$. (*ii*) $\sin^{-1}\frac{x}{5}$.

(*iii*) $\sqrt{2}\sin^{-1}\frac{x}{2a}$ (*iv*) $\frac{1}{a}\sin^{-1}(ax)$.

(*v*) $\sec^{-1}\frac{x}{\sqrt{3}}$. (*vi*) $\sin^{-1}x(x-1)$.

(*vii*) $\sin^{-1}\left(\frac{x+2}{\sqrt{6}}\right)$.

3. (*i*) $\sec^{-1}x$. (*ii*) $\sec^{-1}(3x)$.

(*iii*) $\frac{1}{2}\sec^{-1}(2x)$. (*iv*) $2\sec^{-1}\frac{x}{2}$.

4 (i) $-\cos^{-1} x-\sqrt{1-x^2}$.

(ii) $\sqrt{a^2-x^2}-a\cos^{-1}\left(\dfrac{x}{a}\right)$.

(iii) $\dfrac{(\pi-2)\,a^2}{4}$.

5. (i) $\log(x^3+4)$.

(ii) $\frac{1}{2}\log(3x^2+2x-1)$.

(iii) $\log(\tan^{-1} x)$.

(iv) $\log\tan x$.

(v) $\log(\log x)$.

(vi) $\log(e^x-e^{-x})$.

(vii) $\log(x+\cos x)$.

(viii) $-\log\cot x$.

(ix) $-\dfrac{1}{b}\log(a+b\cos x)$.

(x) $\log(\sin^{-1} x)$.

(xi) $\sin(\tan^{-1} x)$.

(xii) $\tan^{-1}(\sin x)$.

(xiii) $\log 2$.

(xiv) $\log(1+\log x)$.

(xv) $-\frac{1}{2}\log(3\cos^2 x+2\sin^2 x)$.

6. (i) $e^{\tan^{-1} x}$.

(ii) e^x.

(iii) $\dfrac{(\tan^{-1} x)^3}{3}$.

(iv) $\dfrac{1}{1-n}\times\dfrac{1}{(\log x)^{n-1}}$.

(v) $\dfrac{(\log x)^{n-1}}{n+1}$.

(vi) $\dfrac{\sec^3 x}{3}$.

(vii) $\tan^{-1} x^3$.

(viii) $\dfrac{\tan^6 x}{6}$.

(ix) $\log(\log\sin x)$.

(x) $\tan^{-1}(\sin x)$.

(xi) $-\cos(\log x)$.

(xii) $\dfrac{\cos^3 x}{3}-\cos x$.

(xiii) $e^{\tan\theta}$.

(xiv) $\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x$.

(xv) $2\sin\sqrt{x}$.

(xvi) $\dfrac{(3+t^3)^{4/3}}{4}$.

(xvii) $\frac{4}{9}(1+t^3)^{3/2}$.

(xviii) $-\dfrac{\cos^3 x}{3}+\dfrac{\cos^5 x}{5}$.

(xix) $-\dfrac{\cos^3 x}{3}$.

Exercise XX

1. $\dfrac{1}{\sqrt{2}}\log\tan\left(\dfrac{\pi}{8}+\dfrac{x}{2}\right)$.

2. $\dfrac{1}{5}\log\tan\dfrac{1}{2}\left(x+\tan^{-1}\dfrac{4}{3}\right.$.

3. $\frac{1}{13}\log\tan\left[\frac{1}{2}\left(x+\tan^{-1}\frac{12}{5}\right)+\frac{\pi}{4}\right]$.

4. $\sin a\,.\log\sin(x-a)+(x-a)\cos a$.

5. (*i*) $\log\frac{x+\sqrt{x^2+a^2}}{a}$. (*ii*) $\log\frac{x+\sqrt{x^2-a^2}}{a}$.

6. $\frac{1}{2}\tan^{-1}(2\tan x)$.

7. 1. 8. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)$ 9. $\frac{\sqrt{6}\pi}{12}$.

10. $\frac{1}{3}\tan^{-1}\left(\frac{\tan\theta}{3}\right)$. 11. $\frac{\pi}{2\sqrt{5}}$.

12. $-\frac{1}{2}\cdot\frac{1}{2\tan x+1}$

13. $\frac{(1+x^2)^{5/2}}{5}-\frac{2}{3}(1+x^2)^{3/2}+(1+x^2)^{1/2}$.

14. $\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$.

15. $\frac{1}{\sin(a-b)}\log\frac{\sin(x-a)}{\sin(x-b)}$. 16. $\frac{\pi}{4}$.

17. $2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)-\sqrt{2}\log\tan\left(\frac{x}{4}+\frac{\pi}{8}\right)$.

18. $-\frac{2}{b^2}\log(a+b\cos x)-\frac{a}{b^2(a+b\cos x)}$.

19. $\frac{1}{2}[\log(\sec x+\tan x)]^2$.

20. $\frac{1}{2}\log\left(\frac{b}{a}\right)\log(ab)$.

Exercise XXI

1. $\frac{1}{4}\log\frac{x-2}{x+2}$. 2. $\frac{1}{11}\log\frac{2x-1}{3x+4}$.

3. $-3\log(x+1)-\frac{1}{x+1}+4\log(x-2)$.

4. $\frac{1}{3}\tan^{-1}\frac{x}{3}+\frac{1}{5}\tan^{-1}\frac{x}{2}$. 5. $\log\frac{(x+1)(x-1)}{x}$.

6. $\frac{1}{6}\log\frac{1+x^3}{1-x^3}$. 7. $\log(x-1)+\frac{1}{2}\log(x^2+1)+\tan^{-1}x$.

8. $\frac{1}{2}\log(x-1)-\frac{1}{2}\log(x+3)-\frac{1}{x-1}$.

9. $\frac{11}{4}\left\{\log(x+1)-\log(x+3)\right\}+\frac{7}{2}\cdot\frac{1}{x+1}$.

10. $\Sigma \dfrac{a^2}{(a-b)(a-c)} \log (x+a)$. 11. $\log (1+e^{-x})-e^{-x}$. 12. $\dfrac{\pi}{2}$.

13. $\frac{1}{2} \log (1+\cos x)-\frac{1}{6} \log (1-\cos x)-\frac{2}{3} \log (1+\cos 2x)$.

Exercise XXII

1. $(x^3-3x^2+6x-5)e^x$. 2. $\dfrac{x^2}{2} \log x-\dfrac{x^2}{4}$.

3. $-x \cos x+\sin x$. 4. $x \tan x-\log \sec x$.

5. $x \sec x-\log (\sec x+\tan x)$. 6. $x \log x-x$.

7. $\dfrac{x^3}{3} \sin^{-1} x+\frac{1}{3} \sqrt{1-x^2}-\frac{1}{9} (1-x^2)^{3/2}$.

8. $\frac{1}{4} (x^4-1) \tan^{-1} x-\frac{1}{12} (x^3-3x)$.

9. $\frac{1}{8} (\sin 2x-2x \cos 2x)$. 10. $-\cos x \log \cos x+\cos x$.

11. $x (\log x)^2-2x \log x+2x$.

12. $x \sin^{-1} x+\sqrt{1-x^2}$. 13. $x \cosh x-\sinh x$.

14. $\dfrac{x \sqrt{a^2+x^2}}{2}-\dfrac{a^2}{2} \log \dfrac{x+\sqrt{a^2+x^2}}{a}$.

15. $\dfrac{x \sqrt{x^2-a^2}}{2}-\dfrac{a^2}{2} \log \dfrac{x+\sqrt{x^2-a^2}}{a}$.

16. $\dfrac{e^{3x}}{5} \sin (4x-\tan^{-1} \frac{4}{3})$.

17. $e^x \tan^{-1} e^x-\frac{1}{2} \log (1+e^{2x})$.

18. $-\dfrac{\cot^3 x}{3}+\cot x+1$.

19. $\dfrac{1}{a} e^{a \tan^{-1} x}$. 20. $x-\sqrt{1-x^2} \sin^{-1} x$. 21. $e^x \cot \dfrac{x}{2}$.

22. $-\frac{1}{2} \operatorname{cosec} x \cot x+\frac{1}{2} \log \tan \dfrac{x}{2}$.

23. $\dfrac{e^x}{\sqrt{2}}\left\{ x \cos \left(x-\dfrac{\pi}{4}\right)+ \sin \left(x-\dfrac{\pi}{4}\right)\right\}+\frac{1}{2} e^x \sin x$.

Exercise XXIII

1. $\dfrac{b^2-a^2}{2}$. 2. $\dfrac{m^a-m^b}{\log m}$.

3. $\dfrac{b^3-a^3}{3}$. 4. $\cos a-\cos b$.

5. $\dfrac{1}{2}(b-a)+\dfrac{\sin 2a-\sin 2b}{4}$. 6. $\dfrac{b-a}{2}-\dfrac{\sin 2a-\sin 2b}{4}$.

7. $\frac{b^4-a^4}{4}$. 8. 1. 9. $\frac{x^2}{2}$.

10. $\frac{e^{3b}-e^{3a}}{3}$.

Revision Exercise 1

1. $\log a$. 2. 1. 3. $\frac{1}{2}$. 4. 0.
5. $\frac{1}{4}$. 6. 0. 7. 0. 8. 2.
9. 0. 10. 0. 11. -1. 12. e^a.
13. e. 14. $\frac{m}{n}a^{m-n}$. 15. 0. 16. $\frac{2}{3}$.
17. 0. 18. 1. 19. $\frac{1}{2}$. 20. 1.

Revision Exercise 2

1. $\frac{\cos x}{2\sqrt{\sin x}}$. 2. $-3\cos^2 x \sin x$.

3. $\frac{1}{2x\sqrt{\log x}}$. 4. $2x \sec^2 (x^2)$.

5. $\frac{1}{2x}$. 6. $2xe^{x^2}$.

7. $\frac{x}{\sqrt{1+x^2}}$. 8. $\frac{ad-bc}{(cx+d)^2}$.

9. $\frac{1}{2\sqrt{x}(1+x)}$. 10. $\frac{-2x}{\sqrt{1-x^4}}$.

11. $-\frac{1}{x\sqrt{a^2x^2-1}}$. 12. $\frac{-a}{1+(ax+b)^2}$.

13. $\frac{a}{ax+b}$. 14. $-a \operatorname{cosec}^2 (ax+b)$.

15. $x \cos x+\sin x$. 16. $x^2 \sec^2 x+2x \tan x$.

17. $x^{n-1} (1+n \log x)$. 18. $-\frac{1}{2}[3 \sin 3x+\sin x]$.

19. $2 \sin 4x-\sin 2x$. 20. $\frac{\sin x-x \cos x}{\sin^2 x}$.

Revision Exercise 3

1. $\log x$. 2. $\tan^{-1} x$.

3. $\frac{x \log x-(x+1) \log (x+1)}{x(x+1)(\log x)^2}$.

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4. $\frac{\sqrt{a^2-b^2}}{a+b\cos x}$. 96 21 5. $-\frac{1}{2\sqrt{1-x^2}}$.

6. $-\frac{1}{1+x^2}$. 7. $\frac{3x^2}{1+x^6}-\frac{1}{1+x^2}$.

8. $\left(1+\frac{1}{x}\right)^x\left[\log\left(\frac{x+1}{x}\right)-\frac{1}{x+1}\right]+x^{1+1/x}\left[\frac{x+1-\log x}{x^2}\right]$.

9. $\frac{2}{\sqrt{1-x^2}}$. 10. $-\frac{3}{\sqrt{1-x^2}}$. 11. 0.

12. $\frac{y^2}{x(1-y\log x)}$ 13. $\frac{1}{2y-1}$. 14. $\sec x$. 92

15. $\sin^m x\cos^n x[m\cot x-n\tan x]$. 16. $\frac{-1}{2\sqrt{x}(1+x)}$.

17. $(\log x)^{\sin x}\times\left[\frac{\sin x}{x\log x}+\cos x\log\log x\right]+(\sin x)^{\log x}$ 9

$\times\left[\cot x\log x+\frac{\log\sin x}{x}\right]$.

18. $-\frac{(ax+hy+g)}{(hx+by+f)}$. 19. $-\frac{my}{nx}$. 2 19

20. $-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$. 21. $-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$. 22. $\frac{y}{x}$.

Revision Exercise 4

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1. (a) $e^{ax}[xr^n\cos(bx+n\theta)+nr^{n-1}\cos(bx+n\theta-\theta)]$ where $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}(b/a)$.

(b) $\underline{|n}\,[x^2a^n+2nx(ax+b)a^{n-1}+\frac{n(n-1)}{2}(ax+b)^2a^{n-2}]$.

(c) $\frac{xa^n(-1)^{n-1}\underline{|n-1}}{(ax+b)^n}+\frac{na^{n-1}(-1)^{n-2}\underline{|n-2}}{(ax+b)^{n-1}}$.

4.50 (d) $a^{bx}[x^2(b\log a)^n+2nx(b\log a)^{n-1}+n(n-1)(b\log a)^{n-2}]$.

2.50

7.00 (e) $e^x\left[\log x+\frac{^nC_1}{x}-\frac{^nC_2}{x^2}+\dots\dots+\frac{(-1)^{n-1}\underline{|n-1}}{x^n}\right]$.

2. (a) $2^{n/2}e^x\sin\left(x+\frac{n\pi}{4}\right)$. (b) $\frac{5(-1)^{n-1}\underline{|n}\cdot3^{n-1}}{(3x+7)^{n+1}}$.

(c) $\frac{e^x}{4}\left[10^{n/2}\cos(3x+n\tan^{-1}3)+3.2^{n/2}\cos\left(x+\frac{n\pi}{4}\right)\right]$.

(d) $\frac{1}{2}\times5^{n/2}e^x\sin(2x+n\tan^{-1}2)$.

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(e) $\frac{1}{4}\left[6^n \cos\left(6x+\frac{n\pi}{2}\right)+4^n \cos\left(4x+\frac{n\pi}{2}\right)+2^n \cos\left(2x+\frac{n\pi}{2}\right)\right].$

(f) $\frac{1}{8}\left[4.2^n \cos\left(2x+\frac{n\pi}{2}\right)+4^n \cos\left(4x+\frac{n\pi}{2}\right)\right].$

(g) $(-1)^n \underline{|n}\left[\frac{1}{2(x-1)^{n+1}}-\frac{1}{(x-2)^{n+1}}+\frac{1}{2(x-3)^{n+1}}\right].$

(h) $(-1)^n \underline{|n}\left[\frac{1}{(x-3)^{n+1}}-\frac{1}{(x-2)^{n+1}}\right].$

(i) $e^{x \cos \alpha} \sin (x \sin \alpha+n\alpha).$

Revision Exercise 5

1. $\frac{1}{2}[\tan^{-1}(x+1)+\tan^{-1}(x-1)].$
2. $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right).$
3. $\log(1+\sin\theta).$
4. $\frac{e^{2x}}{2}+\frac{a^{2x}}{2 \log a}+\frac{2(ae)^x}{\log(ae)}.$
5. $-\left(e^{-x}+\frac{a^{-x}}{\log a}\right)$
6. $2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right).$
7. $\log \log \log x.$
8. $\frac{1}{\sqrt{2}} \log \tan\left(\frac{\pi}{8}+\frac{\pi}{2}\right).$
9. $\log(\cos x+\sin x).$
10. $\log(\operatorname{cosec}^2 x-\operatorname{cosec} x \cot x).$
11. $\frac{2}{3}(3+x)^{3/2}-2\sqrt{3+x}.$
12. $\frac{2}{3}(x+1)^{3/2}-\frac{2}{3}x^{3/2}.$
13. $\frac{1}{3}\log(x-1)-\frac{1}{2}\log(x-2)+\frac{1}{6}\log(x-4).$
14. $\log x+\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right).$
15. $-\frac{2}{5}\cos^5 x.$
16. $\frac{1}{4}\left[\frac{\cos 6x}{6}-\frac{\cos 4x}{4}-\frac{\cos 2x}{2}\right].$
17. $\log(x^2-2x+2)-\tan^{-1}(x+1).$
18. $\sin^{-1}(2x-1).$
19. $\sin^{-1}(x-2).$
20. $\sec^{-1}(x+1).$
21. $\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+10\log(x-1).$
22. $\tan x-\cot x.$
23. $x+\sin 2x.$
24. $\log \sin(x^2).$
25. $\frac{1}{2}(\log x)^2.$
26. $\sin(e^x).$
27. $\sqrt{1+x^2}.$
28. $\tan\frac{x}{2}+2\log\sec\frac{x}{2}.$

29. $\dfrac{a^x}{\log x}+e^x+\dfrac{x^{a+1}}{a+1}$.

30. $\frac{1}{8}[\sin^{-1}(2x-1)+(2x-1)\sqrt{1-(2x-1)^2}]$.

31. $4\log(x-3)-3\log x(x-2)$.

Revision Exercise 6

1. $m^m.n^n=(m+n)^{m+n}.p^m.\,q^n.\,a^{m+n}$.

4. $a\sin\theta\cos\theta$. 5. $90°$.

6. $(1, 1)$; $2y-3x+1=0$; $3y+2x-5=0$.

7. $y-x+3a=0$; $y+x-3a=0$.

9. $\dfrac{b}{a}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$. 10. $\dfrac{8\pi}{3}$ cubic ft./sec.

11. $\pi/3$ sq. ft./sec. 12. $\frac{3}{2}$ cubic ft./sec.